# Inverse Transient Thermoelastic Problem of Semi-Infinite Thick Hollow Cylinder

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***Abstract-*** **This paper is concerned with inverse transient thermoelastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite hollow cylinder when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.**

***Key Words*****: Semi-infinite hollow cylinder, transient problem, Integral transform, internal heat source, inverse problem**

## I. INTRODUCTION

In 2003, **Noda et al. [1]** have published a book on Thermal Stresses, second edition. **Khobragade [2]** studied Thermoelastic analysis of a thick annular disc with radiation conditions **and Khobragade [3]** discussed Thermoelastic analysis of a thick circular plate**. Pathak et al. [4]** studied Transient Thermo elastic Problems of a Circular Plate with Heat Generation. **Love [5]** published a book on a treatise on the mathematical theory of elasticity. **Marchi and Zgrablich [6]** studies Vibration in hollow circular membrane with elastic supports. **Nowacki [7]** discussed the state of stress in thick circular plate due to temperature field. **Wankhede [8]** studied the quasi-static thermal stresses in a circular plate.

**Ganar et al. [9**] discussed heat transfer and thermal stresses of a thick circular plate. **Singru et al. [10**] studied thermal stress analysis of a thin rectangular plate with internal heat source and Singru [11] discussed thermal stresses of a semi-infinite rectangular slab with internal heat generation. **Pakade et al. [12**] studied transient thermoelastic problem of semi- infinite circular beam with internal heat source. **Lamba et al. [13**] discussed stress functions in a hollow cylinder under heating and cooling processes. **Gahane et al. [14**] studied transient thermoelastic problem of a semi-infinite cylinder with heat sources and **Gahane et al. [15**] discussed thermal stresses in a thick circular plate with internal heat sources. **Hiranwar et al. [16**] studied thermoelastic problem of a cylinder with internal heat sources. **Roy et al. [17**] discussed transient thermoelastic problem of an infinite rectangular slab. **Bagade et al. [18**] studied thermal stresses of a semi infinite rectangular beam.

In this paper, we analyzed inverse thermo elastic problem of temperature and thermal stresses of thick, semi-infinite hollow cylinder due to heat generation. The governing heat conduction equation has been solved by using Marchi-Zgrablich and Fourier Cosine transform techniques. The result presented here will be more useful in engineering applications.

## II. STATEMENT OF THE PROBLEM

Consider a thick hollow cylinder occupying the space D: $a \le r \le b$, $0 \le z < \infty$. The material is homogeneous and isotropic. The differential equation governing the displacement potential function $\phi$ (r ,z ,t) as **Noda et al. [1]** is

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=\left[\frac{1+\upsilon}{1-\upsilon}\right]\alpha_t T \qquad (1)$$

where, $\nu$ and $\alpha_t$ are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the cylinder and T is temperature of the plate satisfying the differential equation as **Noda et al. [1]** is

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\chi(r,z,t)=\frac{1}{k}\frac{\partial T}{\partial t} \qquad (2)$$

Subject to initial condition:

$$T(r,z,0)=f(r,z) \qquad (3)$$

and boundary conditions are

$$\left[T(r,z,t)+k_1\frac{\partial T(r,z,t)}{\partial r}\right]_{r=a}=g_1(z,t) \qquad (4)$$

$$\left[T(r,z,t)+k_2\frac{\partial T(r,z,t)}{\partial r}\right]_{r=\xi}=g_2(z,t) \ \text{(known)} \qquad (5)$$

$$\left[T(r,z,t)\right]_{r=b}=G(z,t) \quad \text{(unknown)} \qquad (6)$$

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=0} = f_1(r,t) \qquad (7)$$

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=\infty} = f_2(r,t) \quad ,0 \le r \le a, \quad t > 0 \qquad (8)$$

where k is the thermal diffusivity of the material of the cylinder.

The displacement function in the cylindrical co-ordinate system are represented by the Goodier thermoelastic function $\phi$ and Love's function L as **Noda et al. [1]** are

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z} \qquad (9)$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2} \qquad (10)$$

in which Goodier thermoelastic potential must satisfy the equation as **Noda et al. [1]** is

$$\nabla^2 \phi = \left(\frac{1+\nu}{1-\nu}\right) a_t T \qquad (11)$$

The Love's function must satisfy

$$\nabla^2(\nabla^2 L) = 0 \qquad (12)$$

Where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

The component of stresses are represented by the use of the potential $\phi$ and Love's function L as **Noda et al.[1]** are

$$\sigma_{rr} = 2G\left\{\left[\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right] + \frac{\partial}{\partial z}\left[\upsilon \nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right]\right\} \qquad (13)$$

$$\sigma_{\theta\theta} = 2G\left\{\left[\frac{1}{r}\frac{\partial \phi}{\partial r} - \nabla^2 \phi\right] + \frac{\partial}{\partial z}\left[\upsilon \nabla^2 L - \frac{1}{r}\frac{\partial^2 L}{\partial r^2}\right]\right\} \qquad (14)$$

$$\sigma_{zz} = 2G\left\{\left[\frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi\right] + \frac{\partial}{\partial z}\left[(2-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right]\right\} \qquad (15)$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left[(1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right]\right\} \qquad (16)$$

Equations (1) to (16) constitute the mathematical formulation of the problem under consideration.

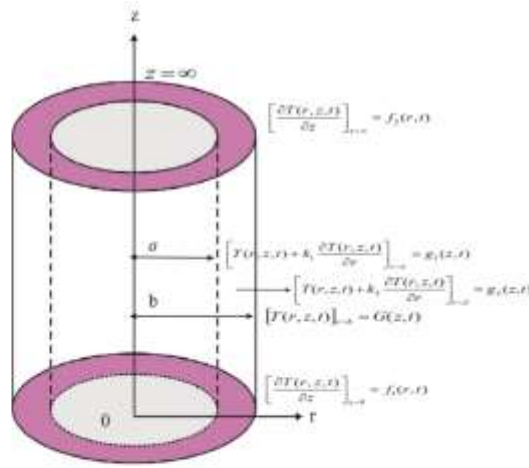


Figure Shows the Geometry of the problem

## III. SOLUTION OF THE PROBLEM

Applying finite **Marchi-Zgrablich transform** defined in [3] to the equations (2) and using equations (4), (5) one obtains

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r,z,t) = \frac{1}{k}\frac{\partial T}{\partial t} \qquad (17)$$

By using the operational property of finite Marchi-Zgrablich transform, we get

$$\frac{\partial^2 \overline{T}}{\partial z^2} - \mu_n^2 \overline{T} + \overline{\chi} = \frac{1}{k}\frac{\partial \overline{T}}{\partial t} + g(z,t) \qquad (18)$$

Again, applying Fourier cosine transform to the equation (18), we get

$$\frac{d\overline{T}_c^*}{dt} + kp^2 \overline{T}_c^* = \phi_1^* + \overline{\chi}_1^* \qquad (19)$$

where

$$\overline{\chi}_1^* = k\overline{\chi}_c^* \quad and \ \phi_1^* = k\mu - k\mu_n^2 \overline{T}_c^* - kg_c^*$$

Equation (19) is a linear equation whose solution is given by

$$\overline{T}^*(n,z,t) = e^{-kp^2 t}\int_0^t \Lambda e^{-kp^2 t'} dt' + Ce^{-kp^2 t} \qquad (20)$$

Where $\Lambda = \left(\phi_1^* + \overline{\chi}_1^*\right)$

Using (3), we get

$$C = F^{*}(m,n)$$

Thus, we have,

$$\overline{T}^{*}(n,z,t) = e^{-kp^2 t}\left[\int_0^t \Lambda e^{-kp^2 t'}dt' + \overline{F}^{*}(m,n)\right] \quad (21)$$

Applying inversion of Fourier cosine transform and Marchi-Zgrablich transform to the equation (21), one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}$$

$$\left\{e^{-kp^2 t}\left[\int_0^t \Lambda e^{-kp^2 t'}dt' + \overline{F}^{*}(m,n)\right]\right\} \times S_0(k_1,k_2,\mu_n r) \quad (22)$$

$$G(z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2 t}\left[\int_0^t \Lambda e^{-kp^2 t'}dt' + \overline{F}^{*}(m,n)\right]\right\} \times S_0(k_1,k_2,\mu_n b) \quad (23)$$

These are the desired solutions of the given problem.

Let us assume Love's function L, which satisfy condition (11) as

$$L(r,z) = \sum_{n=1}^{\infty}\frac{1}{C_n}\psi S_0(k_1,k_2,\mu_n r) \quad (24)$$

where,

$$\psi = e^{-kp^2 t}\left[\int_0^t \Lambda e^{-kp^2 t'}dt' + \overline{F}^{*}(m,n)\right]$$

The displacement potential is given by

$$\phi = A\sum_{n=1}^{\infty}\frac{1}{C_n}\psi S_0(k_1,k_2,\mu_n r)[\psi + B(t)] \quad (25)$$

where,

$$A = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \quad B(t) = e^{-kp^2 t}\left[\int_0^t \Lambda e^{-kp^2 t'}dt' + \overline{F}^{*}(m,n)\right]dt$$

## IV. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting the equations (24) and (25) in the equation (8) one obtains

$$u_r = A\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi + B(t)] - \sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r) \quad (26)$$

$$u_z = 2(1-\upsilon)\left[\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0'(k_1,k_2,\mu_n r) + \frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r)\right] \quad (27)$$

## V. DETERMINATION OF STRESS FUNCTIONS

Substituting the values from the equation (24) and (25) in the equation (10) to (13) we get

$$\sigma_{rr} = 2G\left\{\begin{aligned}&\left[\begin{aligned}&A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi + B(t)]\\&-A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi + B(t)] -\\&\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi + B(t)]\end{aligned}\right]\\&+\frac{\partial}{\partial z}\left[\begin{aligned}&\upsilon\left[\begin{aligned}&A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)\\&+\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r)\end{aligned}\right]\\&-\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)\end{aligned}\right]\end{aligned}\right\} \quad (28)$$

$$\sigma_{\theta\theta} = 2G\left\{\begin{aligned}&\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi + B(t)]\\&\left[\begin{aligned}&A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi + B(t)] +\\&\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi + B(t)]\end{aligned}\right]\\&+\frac{\partial}{\partial z}\left[\begin{aligned}&\upsilon\left[\begin{aligned}&A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0''(k_1,k_2,\mu_n r)\\&+\frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r)\end{aligned}\right]\\&-\frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0''(k_1,k_2,\mu_n r)\end{aligned}\right]\end{aligned}\right\} \quad (29)$$

$$\sigma_{zz} = 2G\left\{\begin{aligned}&\left[\begin{aligned}&A\sum_{n=1}^{\infty}\frac{\mu_n^{\,2}}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi + B(t)]\\&+\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi + B(t)]\end{aligned}\right]\\&+(1-\upsilon)\sum_{n=1}^{\infty}\frac{\mu_n^{\,2}}{C_n}\psi S_0''(k_1,k_2,\mu_n r)\\&+\frac{(1-\upsilon)}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r)\end{aligned}\right\} \quad (30)$$

$$\sigma_{rz} = 2G\left[(1-\upsilon)\sum_{n=1}^{\infty}\frac{\mu_n^{\,3}}{C_n}\psi S_0'''(k_1,k_2,\mu_n r) - \frac{(1-\upsilon)}{r^2}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu_n^{\,2}}{C_n}\psi S_0''(k_1,k_2,\mu_n r)\right] \quad (31)$$

where,

$$A = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \ and$$

$$\psi = e^{-kp^2 t}\left[\int_0^t \Lambda e^{kp^2 t'}dt' + \overline{F}^*(m,n)\right]$$

$$B(t) = \int \psi dt$$

## VI. SPECIAL CASE

Set $F(r, z) = \delta(r - r_0)(z - e^{-z})$ (32)

Applying finite transform defined in Marchi Zgrablich [35] to the equation (32) one obtains

$$\overline{F}(n, z) = r_0\,(z - e^{-z})S_0(k_1, k_2, \mu_n r_0) \quad (33)$$

Substituting the value of (33) in the equations (22) to (23) one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2 t}\left[\int_0^t \Lambda e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]\right\} \times S_0(k_1,k_2,\mu_n r) \quad (34)$$

$$G(z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2 t}\left[\int_0^t \Lambda e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]\right\} \times S_0(k_1,k_2,\mu_n b) \quad (35)$$

where

$$\overline{F}^*(n, m) = r_0\,S_0(k_1, k_2, \mu_n r_0)\int_0^{\infty}(z - e^{-z})\cos\alpha z\,dz$$

## VII. NUMERICAL RESULTS

Put $a = 2, \xi = 2.3, b = 2.5, t = 1\text{sec}, k_1 = 0.25 = k_2$, in equations (34) to (35) one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-15.9 p^2}\left[\int_0^1 \Lambda e^{-15.9 p^2 t'}dt' + \overline{F}^*(m,n)\right]\right\} \times S_0(0.25, 0.25, \mu_n r) \quad (36)$$

$$G(z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2}\left[\int_0^1 \left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]\right\} \times S_0(k_1,k_2,\mu_n(2.5)) \quad (37)$$

## VIII. MATERIAL PROPERTIES

The numerical calculation has been carried out for an Aluminum (pure) circular plate with the material properties as

Density ρ =169 lb/ft$^3$
Specific heat = 0.208 Btu/lbOF
Thermal conductivity K = 15.9 x 10$^6$Btu/(hr. ftOF)
Thermal diffusivity α = 3.33 ft$^2$/hr.
Poisson ratio ν = 0.35
Coefficient of linear thermal expansion $\alpha_t$ = 12.84 x 10$^{-6}$1/F
Lame constantμ = 26.67
Young's modulus of elasticity E = 70G Pa

## IX. DIMENSIONS

The constants associated with the numerical calculation are taken as

Radius of the disk *a* = 2ft

Radius of the disk b = 2.5 ft

## X. CONCLUSION

In this paper, we develop the analysis for the temperature field by introducing the methods of the Marchi- Zgrablich and Fourier cosine transform techniques and determined the expression for temperature distribution, displacement and thermal stresses of a semi-infinite thick hollow cylinder with known boundary conditions which is useful to design of structure or machines in engineering applications.

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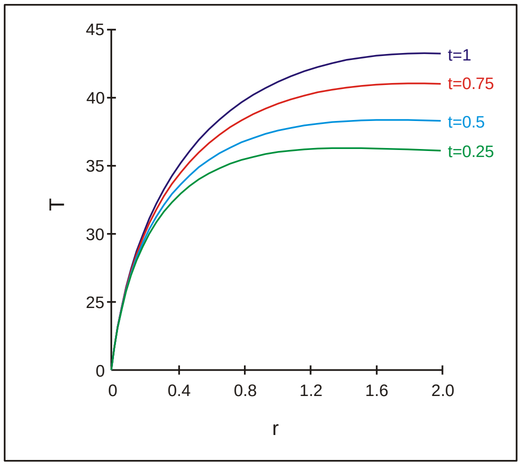


Fig. 1: Temperature distribution vs r

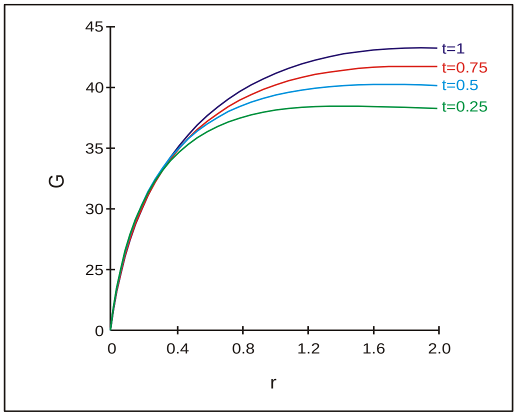


Fig. 2: Unknown Temperature gradient vs r

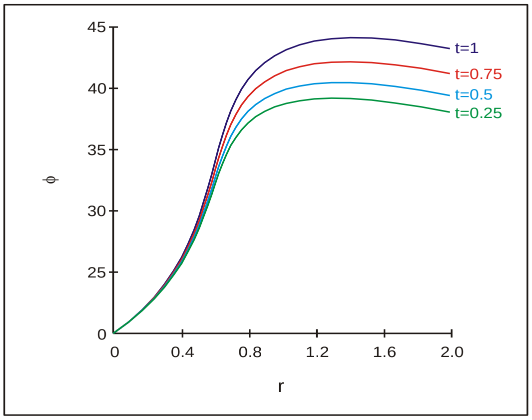


Fig. 3: Displacement function vs r

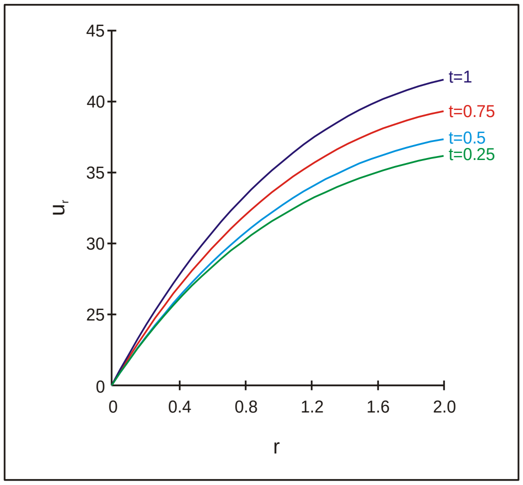


Fig. 4: Displacement component vs r

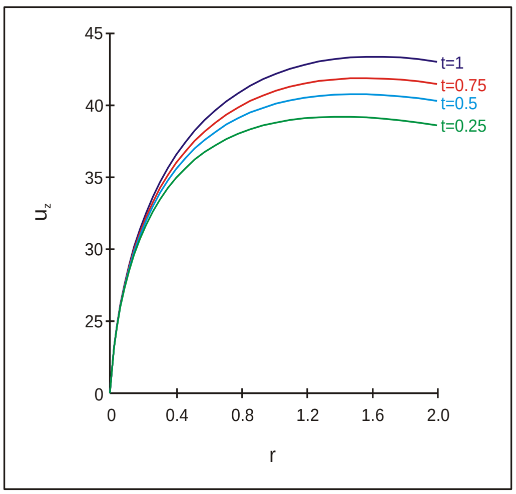


Fig. 5: Displacement component vs r

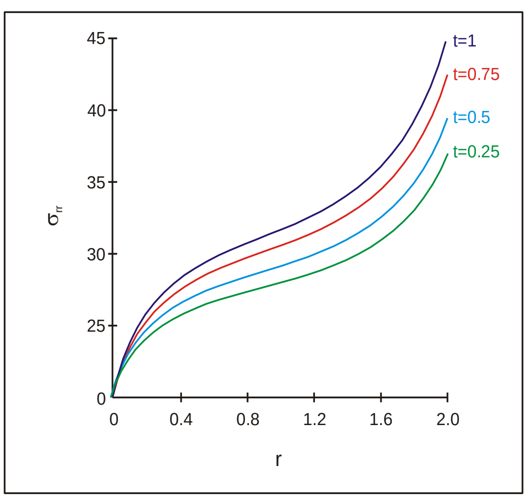


Fig. 6: Thermal stresses vs r

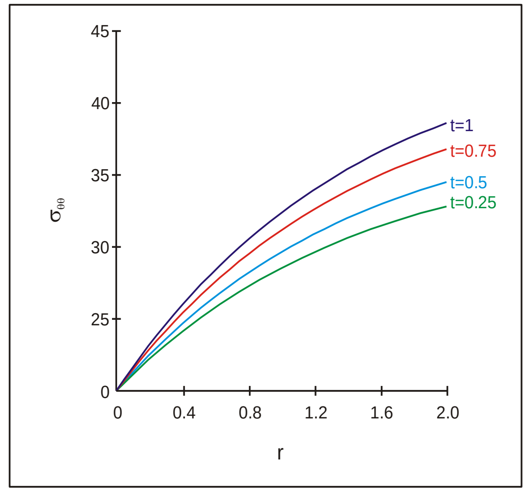


Fig. 7: Thermal stresses vs r

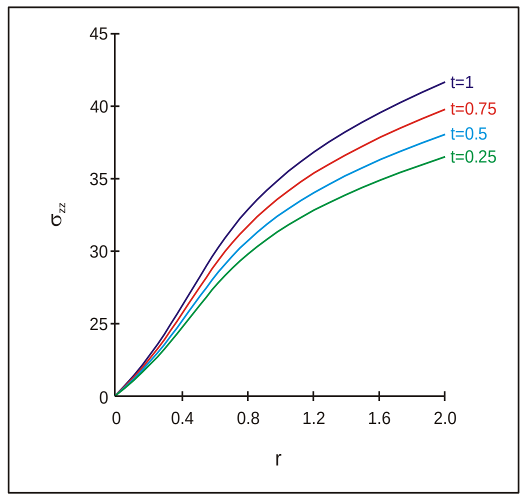


Fig. 8: Thermal stresses vs r

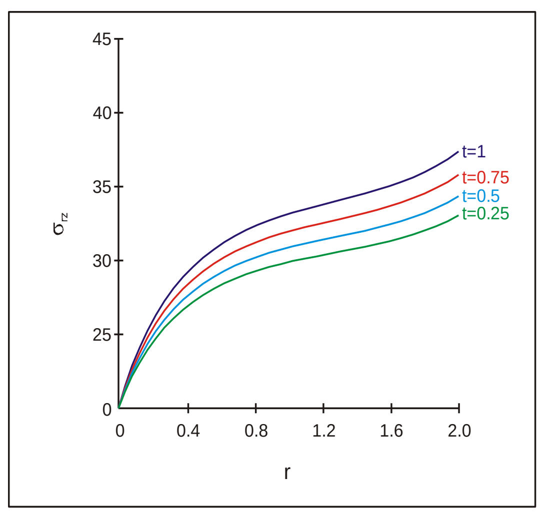


Fig. 9: Thermal stresses vs r