Boundedness of Superposition Operators on the Double Sequence Spaces of Maddox $\mathcal{L}(p)$

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***Abstract*: Sama-ae [18] characterized local boundedness and boundedness of the superposition operator acting from the Maddox sequence space$\boldsymbol{l(p)}$ into $\boldsymbol{l_1}$. Sağır & Güngör [14]defined the superposition operator $\boldsymbol{P_g}$ where $\boldsymbol{g}: \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ by $\boldsymbol{P_g(x) = \left(g(k,s,x_{ks})\right)_{k,s=1}^{\infty}}$ for all real double sequences $\boldsymbol{(x_{ks})}$. The main goal of this paper is constructing the necessary and sufficient conditions for the local boundedness and boundedness of the superposition operator$\boldsymbol{P_g}$acting from Maddox double sequence spaces $\boldsymbol{\mathcal{L}(p)}$ into $\boldsymbol{\mathcal{L}(q)}$ where $\boldsymbol{p = (p_{ks})}$ and $\boldsymbol{q = (q_{ks})}$ is bounded double sequences of positive numbers.**

***Keywords*: Superposition Operators, Local Boundedness, Boundedness, Double Sequence Spaces**

I. INTRODUCTION

Let $\mathbb{R}$ be set of all real numbers, $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Ω denotes the space of all real double sequences which is the vector space with coordinate wise addition and scalar multiplication. Let $x = (x_{ks}) \in \Omega$. If for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{ks} - l| < \varepsilon$ for all $k, s \geq N$, then we call that the double sequence $x = (x_{ks})$ is convergent in the Pringsheim's sense and denoted by $p - \lim x_{ks} = l$. The space of all convergent double sequences in the Pringsheim's sense is denoted by C_p.The space $\mathcal{L}_p$ is defined by

$$\mathcal{L}_p := \left\{ x = (x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^p < \infty \right\}$$

where $1 \leq p < \infty$ and $\sum_{k,s=1}^{\infty} = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty}$ [2,3]. $\mathcal{L}_p$ is a Banach space with the norm

$$\|x\|_p = \left(\sum_{k,s=1}^{\infty} |x_{ks}|^p \right)^{\frac{1}{p}}.$$

The Maddox space $\mathcal{L}(p)$is denoted by

$$\mathcal{L}(p) = \left\{ x = (x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^{p_{ks}} < \infty \right\}$$

Where $p = (p_{ks})$ is a bounded sequence of positive numbers. Let $\|.\|_{\mathcal{L}(p)} : \mathcal{L}(p) \to \mathbb{R}$ be defined by

$$\| x \|_{\mathcal{L}(p)} = \sum_{k,s=1}^{\infty} |x_{ks}|^{\frac{p_{ks}}{M_1}}$$

where $M_1 = \max\left\{1, \sup_{k,s \in \mathbb{N}} p_{ks}\right\}$. We can easily show that $\|x\|_{\mathcal{L}(p)} \geq 0$, $\|x\|_{\mathcal{L}(p)} = 0$ if $x = 0$, $\|x\|_{\mathcal{L}(p)} = \|-x\|_{\mathcal{L}(p)}$ and $\|x + y\|_{\mathcal{L}(p)} \leq \|x\|_{\mathcal{L}(p)} + \|y\|_{\mathcal{L}(p)}$ holds for all $x, y \in \mathcal{L}(p)$. $\|.\|_{\mathcal{L}(p)}$ can not be a norm; however, we can define a metric d on $\mathcal{L}(p)$ by letting $d(x, y) = \| x - y \|_{\mathcal{L}(p)}$ for each $x, y \in \mathcal{L}(p)$[18, 19, 21].

If we consider the sequence(s_{mn}) defined by $s_{mn} = \sum_{k=1}^{m} \sum_{s=1}^{n} x_{ks} (m, n \in \mathbb{N})$, then the pair of $\left((x_{mn}), (s_{mn})\right)$ is called double series. Also(x_{mn}) is called thegeneral term of the series and (s_{mn}) is called the sequence of partial sum. If the sequence of partial sum (s_{mn}) is convergent to a real number s in the Pringsheim's sense, i.e.,

$$p - \lim_{m,n} \sum_{k=1}^{m} \sum_{s=1}^{n} x_{ks} = s$$

Then the series $\left((x_{mn}), (s_{mn})\right)$ is called convergence in the Pringsheim's sense , i.e., p −convergent and the sum of series equal to s, and is denoted by

$$\sum_{k,s=1}^{\infty} x_{ks} = s$$

It is known that if the series is p −convergent, then the p −limit of the general term of the series is zero. There maining term of the series $\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} x_{ks}$ is defined by

$$R_{nm} = \sum_{k=1}^{n-1} \sum_{s=m}^{\infty} x_{ks} + \sum_{k=n}^{\infty} \sum_{s=1}^{m-1} x_{ks} + \sum_{k=n}^{\infty} \sum_{s=m}^{\infty} x_{ks}.$$

and it is demonstrated briefly with

$$\sum_{max\,\{k,s\} \geq N} x_{ks}$$

for $n = m = N$. It is known that if the series is p −convergent, then the p −limit of the remaining term of the series is zero. Once find before mentioned, and more details in

[1,2,3,9,13,20].

Superposition operators on sequence spaces are discussed by some authors. Petranuarat and Kemprasit[11] have characterized continuity of the superposition operator acting from sequences pace l_p into l_q with $1 \leq p, q < \infty$. Sağır and Güngör [14] generalized these works as the superposition operator acting from the space $\mathcal{L}_p$ into $\mathcal{L}_q$ where $1 \leq p, q < \infty$. Sama-ae [18] characterized local boundedness and boundedness of the superposition operator acting from sequence space$l(p)$ into l_1. Formore details see [4,5,6,7,8,10,11,12,14,15,16,17,19]

Let X, Y be two double sequence spaces. A superposition operator P_g on X is a mapping from X into Ω defined by $P_g(x) = \left(g(k, s, x_{ks})\right)_{k,s=1}^{\infty}$where the function $g: \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies

(1) $g(k, s, 0) = 0$ for all $k, s \in \mathbb{N}$.

If $P_g(x) \in Y$ for all $x \in X$, we say that P_g acts from X into Y and write $P_g: X \to Y$ [14]. Moreover, we shall assume the additionally some of the following conditions:

(2)$g(k, s, .)$ is continuous for all $k, s \in \mathbb{N}$

(2′)$g(k, s, .)$ is bounded on every bounded subset of $\mathbb{R}$ for all $k, s \in \mathbb{N}$.

It is obvious that if the function $g(k, s, .)$ satisfies the condition (2), then g satisfies the condition(2′). Also, it is not hard to see that if the function $g(k, s, .)$ is locally bounded on $\mathbb{R}$, then g satisfies condition (2′).

Güngör and Sağır [6] characterized the superposition operator P_g on $\mathcal{L}(p)$ as the following:

Theorem 1. $P_g: \mathcal{L}(p) \to \mathcal{L}(q)$ if and only if there exist $\alpha > 0,\ \beta > 0,\ N \in \mathbb{N}$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that

$$|g(k, s, t)|^{\frac{q_{ks}}{M_2}} c_{ks} + \alpha |t|^{\frac{p_{ks}}{M_1}}$$

whenever$|t| \leq \beta$for all $k, s \in \mathbb{N}$ with $\max\{k, s\} \geq N$.

In this paper, we characterize local boundedness and boundedness of the superposition operator acting from the double sequence space $\mathcal{L}(p)$ into $\mathcal{L}(q)$ where $p = (p_{ks})$ and $q = (q_{ks})$ is bounded double sequences of positive numbers.

II. MAIN RESULTS

Theorem 2. If $P_g: \mathcal{L}(p) \to \mathcal{L}(q)$, then P_g is locally bounded on $\mathcal{L}(p)$ if and only if g satisfies (2′).

Proof. Assume that g satisfies (2′) and let $z = (z_{ks}) \in \mathcal{L}(p)$. There exist $N' \in \mathbb{N}, \alpha, \beta > 0$ and $(c_{ks})_{k_z,s=1}^{\infty} \in \mathcal{L}_1$ such that

$$|g(k, s, t)|^{\frac{q_{ks}}{M_2}} c_{ks} + \alpha |t|^{\frac{p_{ks}}{M_1}} \text{whenever} |t| \leq \beta \qquad (1)$$

foreach $k, s \in \mathbb{N}$ with $\max\{k, s\} \geq N'$. Let $x = (x_{ks}) \in \mathcal{L}(p)$ satisfying $\| z - x \|_{\mathcal{L}(p)} \leq \frac{\beta^{\frac{p_{ks}}{M_1}}}{2}$. So, we have that

$$\sum_{max\,\{k,s\} \geq N} |z_{ks} - x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \frac{\beta^{\frac{p_{ks}}{M_1}}}{2}. \qquad (2)$$

Since $z = (z_{ks}) \in \mathcal{L}(p)$, there exists $N \in \mathbb{N}$ with $N \geq N'$

$$\sum_{\max\,\{k,s\} \geq N} |z_{ks}|^{\frac{p_{ks}}{M_1}} \leq \frac{\beta^{\frac{p_{ks}}{M_1}}}{2}. \qquad (3)$$

Byusing (2) and (3), wefind

$$\sum_{\max\,\{k,s\} \geq N} |x_{ks}|^{\frac{p_{ks}}{M_1}}$$

$$\leq \sum_{\max\,\{k,s\} \geq N} |z_{ks} - x_{ks}|^{\frac{p_{ks}}{M_1}} + \sum_{\max\,\{k,s\} \geq N} |z_{ks}|^{\frac{p_{ks}}{M_1}}$$

$$\leq \beta^{\frac{p_{ks}}{M_1}}$$

For all $k, s \in \mathbb{N}$ with $\max\{k, s\} \geq N$. We obtain that $|x_{ks}| \leq \beta$, hence we can write

$$\sum_{\max\,\{k,s\} \geq N} |g(k, s, x_{ks})|^{\frac{q_{ks}}{M_2}}$$

$$\leq \sum_{\max\,\{k,s\} \geq N} c_{ks} + \alpha \sum_{\max\,\{k,s\} \geq N} |x_{ks}|^{\frac{p_{ks}}{M_1}} \qquad (4)$$

forall $k, s \in \mathbb{N}$ with $\max\{k, s\} \geq N$ by using (1). Let

$$m_{ks} = \sup_{|t - z_{ks}| \leq \frac{\beta^{\frac{}{M_1}}}{2^{p_{ks}}}} |g(k, s, t)|^{\frac{q_{ks}}{M_2}}.$$

Since g satisfies condition(2′) we see that $m_{ks} < \infty$ for all $k, s \in \mathbb{N}$. We have

$$|g(k, s, x_{ks})|^{\frac{q_{ks}}{M_2}} \leq m_{ks} \qquad (5)$$

forall $k, s \in \mathbb{N}$. By using (4) and (5), we obtain

$$\|P_g(x)\|_{\mathcal{L}(q)} = \sum_{k,s=1}^{\infty} |g(k, s, x_{ks})|^{\frac{q_{ks}}{M_2}}$$

$$= \sum_{k,s=1}^{N-1} |g(k, s, x_{ks})|^{\frac{q_{ks}}{M_2}} + \sum_{\max\,\{k,s\} \geq N} |g(k, s, x_{ks})|^{\frac{q_{ks}}{M_2}}$$

$$\leq \sum_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1 + \alpha \beta^{\frac{p_{ks}}{M_1}} < \infty.$$

If we take $A = \sum_{k,s=1}^{N-1} m_{ks} + \|c_{ks}\|_1 + \alpha \beta^{\frac{p_{ks}}{M_1}} < \infty$, then

$$\|P_g(x) - P_g(z)\|_{\mathcal{L}(q)} \leq \|P_g(x)\|_{\mathcal{L}(q)} + \|P_g(z)\|_{\mathcal{L}(q)} \leq \|P_g(z)\|_{\mathcal{L}(q)} + A.$$

Let$\gamma = \|P_g(z)\|_{\mathcal{L}(q)} + A$, then we write $\|P_g(x) - Pgz\mathcal{L}(q) \leq \gamma$. Hence, Pg is locally bounded on $\mathcal{L}(p)$.

Conversely, assume that P_g is locally bounded on $\mathcal{L}(p)$. To complete the proof, it is sufficient that g is locally bounded on $\mathbb{R}$. Let $k, s \in \mathbb{N}$ and $a \in \mathbb{R}$. The sequence $y = (y_{ks})$ is defined as

$$y_{ks} = \begin{cases} a & , k = n \quad \text{and} \quad s = m \\ 0 & , \text{others} \end{cases}$$

So, it is obvious that $y = (y_{ks}) \in \mathcal{L}(p)$. From the hypothesis, there exist $\alpha, \beta > 0$ such that

$$\|P_g(x) - P_g(y)\|_{\mathcal{L}(q)} \leq \beta \text{ whenever } \|x - y\|_{\mathcal{L}(p)} \leq \alpha. \qquad (6)$$

Also, the sequence $x = (x_{ks})$ is defined as

$$x_{ks} = \begin{cases} b & , k = n \quad \text{and} \quad s = m \\ 0 & , \text{others} \end{cases}$$

forall $k, s \in \mathbb{N}$ and $b \in \mathbb{R}$ with $|b - a| \leq \alpha^{\frac{M_1}{p_{ks}}}$. So, it is obvious that $x = (x_{ks}) \in \mathcal{L}(p)$. Hence, we find

$$\| x - y \|_{\mathcal{L}(p)} = \sum_{k,s \in \mathbb{N}} |x_{ks} - y_{ks}|^{\frac{p_{ks}}{M_1}} = |b - a|^{\frac{p_{ks}}{M_1}} \leq \alpha.$$

Therefore, $\|P_g(x) - P_g(y)\|_{\mathcal{L}(q)} \leq \beta$ from (6). Then, we obtain

$$|g(k,s,b) - g(k,s,a)| \leq \left(\sum_{k,s=1}^{\infty} |g(k,s,x_{ks}) - g(k,s,y_{ks})|^{\frac{q_{ks}}{M_2}} \right)^{\frac{M_2}{q_{ks}}} = \|P_g(x) - P_g(y)\|_{\mathcal{L}(q)}^{\frac{M_2}{q_{ks}}} \leq \beta^{\frac{M_2}{q_{ks}}}$$

Since $b \in \mathbb{R}$ is arbitrary, $g(k,s,.)$ is locally bounded on $\mathbb{R}$.

Theorem 3. If $P_g: \mathcal{L}(p) \to \mathcal{L}(q)$, then P_g is bounded on $\mathcal{L}(p)$ if and only if for every $\beta > 0$ there exist $\alpha(\beta) > 0$ and a sequence $c(\beta) = (c_{ks}(\beta)) \in \mathcal{L}_1$ such that

$$|g(k,s,t)|^{\frac{q_{ks}}{M_2}} \leq c_{ks}(\beta) + \frac{2\alpha(\beta)}{\beta} |t|^{\frac{p_{ks}}{M_1}} \qquad (7)$$

whenever $|t| \leq \beta$foreach $k, s \in \mathbb{N}$.

Proof. Suppose that the inequality (7) holds. Let $\beta > 0$ and $x = (x_{ks}) \in \mathcal{L}(p)$ with $\|x_{ks}\|_{\mathcal{L}(p)} \leq \beta^{\frac{p_{kx}}{M_1}}$. Thus, we can write $\sum_{k,s=1}^{\infty} |x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \beta^{\frac{p_{ks}}{M_1}}$, that is, $|x_{ks}| \leq \beta$ for all $k, s \in \mathbb{N}$. By assumption, there are $\alpha(\beta) > 0$ and a sequence $c(\beta) = (c_{ks}(\beta)) \in \mathcal{L}_1$ such that $|g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} \leq c_{ks}(\beta) + \frac{2\alpha(\beta)}{\beta} |x_{ks}|^{\frac{p_{ks}}{M_1}}$. Since $P_g(x) = (g(k,s,x_{ks})) \in \mathcal{L}(q)$, we find

$$\|P_g(x)\|_{\mathcal{L}(q)} = \sum_{k,s=1}^{\infty} |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}}$$
$$\leq \sum_{k,s=1}^{\infty} c_{ks}(\beta) + \frac{2\alpha(\beta)}{\beta} \sum_{k,s=1}^{\infty} |x_{ks}|^{\frac{p_{ks}}{M_1}}$$
$$\leq \sum_{k,s=1}^{\infty} c_{ks}(\beta) + \frac{2\alpha(\beta)}{\beta} \beta^{\frac{p_{ks}}{M_1}}$$
$$\leq \sum_{k,s=1}^{\infty} |c_{ks}(\beta)| + 2\alpha(\beta)\beta^{\frac{p_{ks}}{M_1} - 1}.$$

Therefore, P_g is bounded.

Assume that the P_g acting from $\mathcal{L}(p)$ to $\mathcal{L}(q)$ is bounded. Let $\beta > 0$. For each bounded double sequence $x = (x_{ks})$ with $\| x \|_{\mathcal{L}(p)} \leq \beta$, we have that $\sum_{k,s=1}^{\infty} |x_{ks}|^{\frac{p_{ks}}{M_1}} \leq \beta$. Since P_g is bounded, there exists $\alpha(\beta) > 0$ such that $\|P_g(x)\|_{\mathcal{L}(q)} \leq \alpha(\beta) < \infty$. Also, g satisfies condition (2′).We define

$$h_\beta(k,s,t) = max\left\{0, |g(k,s,t)|^{\frac{q_{ks}}{M_2}} - \frac{2\alpha(\beta)}{\beta} |t|^{\frac{p_{ks}}{M_1}}\right\}$$

and

$$c_{ks}(\beta) = sup\{h_\beta(k,s,t) : |t| \leq \beta\}$$

foreach $k \in \mathbb{N}$. Let $k, s \in \mathbb{N}$ and $t \in \mathbb{R}$ with $|t| \leq \beta$. It is obvious that $|g(k,s,t)|^{\frac{q_{ks}}{M_2}} \leq c_{ks}(\beta) + \frac{2\alpha(\beta)}{\beta} |t|^{\frac{p_{ks}}{M_1}}$ if $h_\beta(k,s,t) = 0$. Suppose that $h_\beta(k,s,t) \neq 0$, hence we get $h_\beta(k,s,t) = |g(k,s,t)|^{\frac{q_{ks}}{M_2}} - \frac{2\alpha(\beta)}{\beta} |t|^{\frac{p_{ks}}{M_1}}$. Therefore, wefind $|g(k,s,t)|^{\frac{q_{ks}}{M_2}} \leq c_{ks}(\beta) + \frac{2\alpha(\beta)}{\beta} |t|^{\frac{p_{ks}}{M_1}}$ when $|t| \leq \beta$. We will show that $(c_{ks}(\beta)) \in \mathcal{L}_1$. Since $g$satisfies condition(2′), $h_\beta(k,s,t)$ is bounded on every bounded subset of real numbers for all $k, s \in \mathbb{N}$. Then we have that $0 \leq c_{ks}(\beta) < \infty$ for all $k, s \in \mathbb{N}$. By definition of $c_{ks}(\beta)$, we find that for each $\varepsilon > 0$, there exists the sequence $y = (y_{ks})$ with $|y_{ks}| \leq \beta$ for all $k, s \in \mathbb{N}$ and

$$c_{ks}(\beta) < h_\beta(k,s,y_{ks}) + \frac{\varepsilon}{2^{k+s}}.$$

We define the sequence$y' = (y'_{ks})$suchthat

$$y'_{ks} = \begin{cases} y_{ks}, & h_\beta(k,s,t) > 0 \\ 0, & h_\beta(k,s,t) = 0 \end{cases}$$

We can find finite sequences (m_i) and (n_j) with $m_1 = 1 <$

$m_2 < \cdots < m_r = m$ and $n_1 = 1 < n_2 < \cdots < n_r = n$ for any $m, n \in \mathbb{N}$ such that

$$\sum_{k=1}^{m}\sum_{s=1}^{n} |y'_{ks}|^{\frac{p_{k_s}}{M_1}}$$

$$= \sum_{i=1}^{r-2}\sum_{j=1}^{r-2}\left(\sum_{k=m_i}^{m_{i+1}-1}\sum_{s=n_j}^{n_{j+1}-1} |y'_{ks}|^{\frac{p_{ks}}{M_1}}\right) + \sum_{k=m_{r-1}}^{m}\sum_{s=n_{r-1}}^{n} |y'_{ks}|^{\frac{p_{ks}}{M_1}}$$

with $\frac{\beta}{2} \le \sum_{k=m_i}^{m_{i+1}-1}\sum_{s=n_j}^{n_{j+1}-1} |y'_{ks}|^{\frac{p_{ks}}{M_1}} \le \beta$ for $i, j = 1,2,\dots, r-2$ and $0 \le \sum_{k=m_{r-1}}^{m}\sum_{s=n_{r-1}}^{n} |y'_{ks}|^{\frac{p_{ks}}{M_1}} \le \beta$. For each $i, j = 1,2,\dots, r-2$, we set

$$z^{(ij)} = \begin{cases} y'_{ks}, & k \in \{m_i, m_i+1, \dots, m_{i+1}-1\} \text{ and } s \in \{n_j, n_j+1, \dots, n_{j+1}-1\} \\ 0, & \text{other} \end{cases}$$

$$z^{((r-1)(r-1))} = \begin{cases} y'_{ks}, & k \in \{m_{r-1}, m_{r-1}+1, \dots, m_r = m\} \text{ and } s \in \{n_{r-1}, n_{r-1}+1, \dots, n_r = n\} \\ 0, & \text{other} \end{cases}$$

We see that $z^{(ij)}$ is a bounded double sequence and $\sum_{k,s=1}^{\infty} \left|z_{ks}^{(ij)}\right|^{\frac{p_{ks}}{M_1}} \le \beta$ for each $i, j = 1,2,\dots, r-1$. Since P_g is bounded, we obtain that $\sum_{k,s=1}^{\infty} \left|g\left(k,s,z_{ks}^{(ij)}\right)\right| \le \alpha(\beta)$ for all $i, j = 1,2,\dots, r-1$. Thus,

$$\sum_{k=m_i}^{m_{i+1}-1}\sum_{s=n_j}^{n_{j+1}-1} \left|g\left(k,s,z_{ks}^{(ij)}\right)\right| \le \alpha(\beta) \qquad (8)$$

for all $i, j = 1,2,\dots, r-2$ and

$$\sum_{k=m_{r-1}}^{m}\sum_{s=n_{r-1}}^{n} \left|g\left(k,s,z_{ks}^{(ij)}\right)\right| \le \alpha(\beta). \qquad (9)$$

We define

$$g'(k,s,t) = \begin{cases} g(k,s,t) & , h_\beta(k,s,t) > 0 \\ 0 & , h_\beta(k,s,t) = 0 \end{cases}.$$

We find that

$$\sum_{k=m_i}^{m_{i+1}-1}\sum_{s=n_j}^{n_{j+1}-1} \left|g'\left(k,s,z_{ks}^{(ij)}\right)\right| \le \alpha(\beta)$$

for all $i, j = 1,2,\dots, r-2$ and

$$\sum_{k=m_{r-1}}^{m}\sum_{s=n_{r-1}}^{n} \left|g'\left(k,s,z_{ks}^{(ij)}\right)\right| \le \alpha(\beta)$$

By using (8) and (9). Therefore, we obtain that

$$h_\beta(k,s,y_{ks}) = |g'(k,s,y'_{ks})|^{\frac{q_{ks}}{M_2}} - \frac{2\alpha(\beta)}{\beta}|y'_{ks}|^{\frac{p_{ks}}{M_1}}. \qquad (10)$$

Byusing (10), we get that

$$\sum_{k=1}^{m}\sum_{s=1}^{n} c_{ks}(\beta)$$

$$< \sum_{k=1}^{m}\sum_{s=1}^{n} h_\beta(k,s,y_{ks}) + \sum_{k=1}^{m}\sum_{s=1}^{n} \frac{\varepsilon}{2^{k+s}}$$

$$= \sum_{i=1}^{r-2}\sum_{j=1}^{r-2}\left(\sum_{k=m_i}^{m_{i+1}-1}\sum_{s=n_j}^{n_{j+1}-1} h_\beta(k,s,y_{ks})\right) + \sum_{k=m_{r-1}}^{m}\sum_{s=n_{r-1}}^{n} h_\beta(k,s,y_{ks}) + \sum_{k=1}^{m}\sum_{s=1}^{n} \frac{\varepsilon}{2^{k+s}}$$

$$= \sum_{i=1}^{r-2}\sum_{j=1}^{r-2}\left[\sum_{k=m_i}^{m_{i+1}-1}\sum_{s=n_j}^{n_{j+1}-1}\left(|g'(k,s,y'_{ks})|^{\frac{q_{ks}}{M_2}} - \frac{2\alpha(\beta)}{\beta}|y'_{ks}|^{\frac{p_{ks}}{M_1}}\right)\right] + \sum_{k=m_{r-1}}^{m}\sum_{s=n_{r-1}}^{n}\left(|g'(k,s,y'_{ks})|^{\frac{q_{ks}}{M_2}} - \frac{2\alpha(\beta)}{\beta}|y'_{ks}|^{\frac{p_{ks}}{M_1}}\right) + \sum_{k=1}^{m}\sum_{s=1}^{n} \frac{\varepsilon}{2^{k+s}}$$

$$= \sum_{i=1}^{r-2}\sum_{j=1}^{r-2}\left[\sum_{k=m_i}^{m_{i+1}-1}\sum_{s=n_j}^{n_{j+1}-1}\left(\left|g'\left(k,s,z_{ks}^{(ij)}\right)\right|^{\frac{q_{ks}}{M_2}} - \frac{2\alpha(\beta)}{\beta}\left|z_{ks}^{(ij)}\right|^{\frac{p_{ks}}{M_1}}\right)\right] + \sum_{k=m_{r-1}}^{m}\sum_{s=n_{r-1}}^{n}\left(\left|g'\left(k,s,z_{ks}^{((r-1)(r-1))}\right)\right|^{\frac{q_{ks}}{M_2}} - \frac{2\alpha(\beta)}{\beta}\left|z_{ks}^{((r-1)(r-1))}\right|^{\frac{p_{ks}}{M_1}}\right) + \sum_{k=1}^{m}\sum_{s=1}^{n} \frac{\varepsilon}{2^{k+s}}$$

$$= (r-2)^2\alpha(\beta) + \alpha(\beta) - \frac{2\alpha(\beta)}{\beta}(r-2)^2\frac{\beta}{2}$$

$$+\sum_{k=1}^{m}\sum_{s=1}^{n}\frac{\varepsilon}{2^{k+s}}$$

$$\leq \alpha(\beta)+\sum_{k=1}^{m}\sum_{s=1}^{n}\frac{\varepsilon}{2^{k+s}}$$

Then, $\sum_{k,s=1}^{\infty} c_{ks}(\beta)=\lim_{m,n\to\infty}\sum_{k=1}^{m}\sum_{s=1}^{n} c_{ks}(\beta)\leq\alpha(\beta)+\varepsilon$. Since $\varepsilon>0$ is arbitrary, $\sum_{k,s=1}^{\infty} c_{ks}(\beta)\leq\alpha(\beta)$. Hence we get $\|c_{ks}(\beta)\|_1\leq\alpha(\beta)$. The proof is completed.

Example 1. Let $g:\mathbb{N}^2\times\mathbb{R}\to\mathbb{R}$ be defined by

$$g(k,s,t)=\left(\left(\frac{1}{3^{k+s}}+|t|^{\frac{p_{ks}}{M_1}}\right)|t|^{p_{ks}}\right)^{\frac{M_2}{q_{ks}}}$$

For all $k,s\in\mathbb{N}$ and $t\in\mathbb{R}$. Since g satisfies condition $(2')$, $P_g$is locally bounded on $\mathcal{L}(p)$from Theorem 2. Let$\beta>0$ and $t\in\mathbb{R}$ with $|t|\leq\beta$. For each $k,s\in\mathbb{N}$,

$$|g(k,s,t)|^{\frac{q_{ks}}{M_2}}=\left(\frac{1}{3^{k+s}}+|t|^{\frac{p_{ks}}{M_1}}\right)|t|^{p_{ks}}$$

$$=\frac{1}{3^{k+s}}|t|^{p_{ks}}+|t|^{\frac{p_{ks}}{M_1}}|t|^{p_{ks}}$$

$$\leq\frac{\beta^{M_1}}{3^{k+s}}+\beta^{M_1}|t|^{\frac{p_{ks}}{M_1}}.$$

Let $\alpha(\beta)=\frac{\beta^{M_1+1}}{2}$ and $c_{ks}(\beta)=\frac{\beta^{M_1}}{3^{k+s}}$ for all $k,s\in\mathbb{N}$. Therefore, $c_{ks}(\beta)\in\mathcal{L}_1$and $|g(k,s,t)|\leq c_{ks}(\beta)+2\frac{\alpha(\beta)}{\beta}|t|^{p_{ks}}$. By Theorem 3, P_g is bounded on $\mathcal{L}(p)$.

Example 2. Let $g:\mathbb{N}^2\times\mathbb{R}\to\mathbb{R}$ be defined by

$$g(k,s,t)=\left(\frac{|t|}{k^4+s^4}+2|t|^{\frac{p_{ks}}{M_1}+1}\right)^{\frac{M_2}{q_{ks}}}$$

For all $k,s\in\mathbb{N}$ and $t\in\mathbb{R}$. Since g satisfies condition$(2')$, $P_g$is locally bounded on $\mathcal{L}(p)$ from Theorem 2. Let $\beta>0$ and $t\in\mathbb{R}$ with $|t|\leq\beta$. For each $k,s\in\mathbb{N}$,

$$|g(k,s,t)|^{\frac{q_{ks}}{M_2}}=\frac{|t|}{k^4+s^4}+2|t||t|^{\frac{p_{ks}}{M_1}}$$

$$\leq\frac{\beta}{k^4+s^4}+2\beta|t|^{\frac{p_{ks}}{M_1}}.$$

Let $\alpha(\beta)=\beta^2$ and $c_{ks}(\beta)=\frac{\beta}{k^4+s^4}$ for all $k,s\in\mathbb{N}$. Therefore, $c_{ks}(\beta)\in\mathcal{L}_1$ and $|g(k,s,t)|\leq c_{ks}(\beta)+2\frac{\alpha(\beta)}{\beta}|t|^{p_{ks}}$. By Theorem 3, P_g is bounded on $\mathcal{L}(p)$.

III. CONCLUSIONS

In this paper, the necessary and sufficient conditions for local boundedness and boundedness of the superposition operator acting from the double sequence space $\mathcal{L}(p)$ into $\mathcal{L}(q)$ where $p=(p_{ks})$ and $q=(q_{ks})$ is bounded double sequences of positive numbers are obtained.

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