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On Characterization of *-Boundedness of Superposition Operators on the Sequence Spaces $\ell_{p,\alpha}$

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***Abstract*: There have been many researching about of non-Newtonian calculus and superposition operators until present. The non-Newtonian superposition operator was introduced by Sağır and Erdoğan in [15]. In this study, *-locally boundedness and *-boundedness of the non-Newtonian superposition operator $_{N}P_f : \ell_{p,\alpha} \rightarrow \ell_{1,\beta}$ were characterized.**

***Keywords*: *-boundedness, *-locally boundedness, non-Newtonian calculus, non-Newtonian superposition operators, non-Newtonian sequence spaces.**

I. INTRODUCTION

Non-Newtonian calculus was firstly defined by Michael Grossman and Robert Katz between years 1967 and 1970. They published the book about fundamentals of non-Newtonian calculus which includes some special calculus such as geometric, harmonic, quadratic. At the recent times, Duyar and Erdogan [10] studied on non-Newtonian real number series and there are many works about non-Newtonian calculus as [8,12,13,16,18].

Many studies are done until today on superposition operator which is one of the non-linear operators. Dedagich and Zabreiko [2] worked on the superposition operators in the space ℓ_p. After, some properties of superposition operator, such as boundedness, continuity, were studied by Kolk and Raidjoe [5], Tainchai [3] and many others [4,7,9,11,14]. Non-Newtonian superposition operator was defined and characterized in some non-Newtonian sequence spaces by Sağır and Erdoğan [15] . Also Erdoğan and Sağır [17] worked on *-boundedness of some non-Newtonian superposition operators.

Our purpose in this paper is to extend some topological properties of superposition operators in classical calculus to *-calculus. So we obtained that the necessary and sufficient conditions for the *- boundedness and *-locally boundedness of the non-Newtonian superposition operator $_{N}P_f : \ell_{p,\alpha} \rightarrow \ell_{1,\beta}$ in here.

A *generator* is defined as one-to-one function with domain $\mathbb{R}$ and the range of generator is a subset of $\mathbb{R}$. Let take any α generator with range $A = \mathbb{R}_\alpha$. Let define α-addition, α- subtraction, α-multiplication, α-division and α-order as follows respectively;

$$y \dot{+} z = \alpha\left(\alpha^{-1}(y) + \alpha^{-1}(z)\right)$$
$$y \dot{-} z = \alpha\left(\alpha^{-1}(y) - \alpha^{-1}(z)\right)$$
$$y \dot{\times} z = \alpha\left(\alpha^{-1}(y) \times \alpha^{-1}(z)\right)$$
$$y \dot{/} z = \alpha\left(\alpha^{-1}(y) / \alpha^{-1}(z)\right), z \neq \dot{0}, \alpha^{-1}(z) \neq 0$$
$$y \dot{<} z \, (y \dot{\leq} z) \Leftrightarrow \alpha^{-1}(y) < \alpha^{-1}(z) \left(\alpha^{-1}(y) \leq \alpha^{-1}(z)\right)$$

for $x, y \in \mathbb{R}_\alpha$ [1].

$\left(\mathbb{R}_\alpha, \dot{+}, \dot{\times}, \dot{\leq}\right)$ is totally ordered field [6].

Grossman and Katz described the *-calculus with the help of two arbitrary selected generators. In this study, we studied according to *-calculus. Let take any generators α and β and let * ("star") is shown the ordered pair of arithmetics (α-arithmetic, β-arithmetic). The following notations will be used.

	α-arithmetic	β-arithmetic
Realm	$A = \mathbb{R}_\alpha$	$B = \mathbb{R}_\beta$
Addition	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Order	$\dot{<}$	$\ddot{<}$

In the *-calculus, α -arithmetic is used on arguments and β -arithmetic is used on values.

The isomorphism from α -arithmetic to β -arithmetic is the unique function ι (iota) that possesses the following three properties.

1. ι is one-to-one.

2. ι is on A and onto B.

3. For any numbers u and v in A,

It turns out that $\iota(x)=\beta(\alpha^{-1}(x))$ for every number x in A and that $\iota(\dot{n})=\ddot{n}$ for every integer n [1].

The non-Newtonian sequence spaces S_α and $\ell_{p,\alpha}$ over the non-Newtonian real field $□_\alpha$ are defined as following:

$$S_\alpha=\left\{x=(x_k):\forall k\in □;\ x_k\in □_\alpha\right\}$$

$$\ell_{p,\alpha}=\left\{x=(x_k)\in S_\alpha:\ {}_\alpha\sum_{k=1}^{\infty}|x_k|_\alpha^{p_\alpha}\,\dot{<}\,\dot{+}\infty\right\}\quad(1\le p<\infty).$$

The sequence space $\ell_{p,\alpha}$ is non-Newtonian normed spaces with the non-Newtonian norm $\|.\|_{\ell_{p,\alpha}}$ which is defined as $\|x\|_{\ell_{p,\alpha}}=\left({}_\alpha\sum_{k=1}^{\infty}|x_k|_\alpha^{p_\alpha}\right)^{(1/p)_\alpha}$ [6].

Let S_α be space of non-Newtonian real number sequences, X_α be a sequence space on $□_\alpha$ and Y_β be a sequence space on $□_\beta$. A non-Newtonian superposition operator ${}_NP_f$ on X_α is a mapping from X_α into S_α defined by ${}_NP_f(x)=\left(f(k,x_k)\right)_{k=1}^{\infty}$ where $f:□\times□_\alpha\to□_\beta$ satisfies condition (NA_1) as follows;

(NA_1) $f(k,\dot{0})=\ddot{0}$ for all $k\in□$

If ${}_NP_f\in Y_\beta$ for all $x=(x_k)\in X_\alpha$, we say that ${}_NP_f$ acts from X_α into Y_β and write ${}_NP_f:X_\alpha\to Y_\beta$ [15].

Also, we shall assume the following conditions:

(NA_2) $f(k,.)$ is *-continuous for all $k\in□$.

$(NA_2^{'})$ $f(k,.)$ is β -bounded on every α -bounded subset of $□_\alpha$ for all $k\in□$.

Sağır and Erdoğan [15] have characterized the non-Newtonian superposition operators ${}_NP_f$ on $\ell_{p,\alpha}$ as the following.

Theorem 1: Let $f:□\times□_\alpha\to□_\beta$ satisfies the condition $(NA_2^{'})$. Then ${}_NP_f:\ell_{p,\alpha}\to\ell_{1,\beta}$ if and only if there exist α -numbers $\mu,\eta\,\dot{>}\,\dot{0}$ and a β -sequence $(c_k)\in\ell_{1,\beta}$ such that

$$|f(k,t)|_\beta\,\ddot{\le}\,c_k\,\ddot{+}\,\iota(\eta)\,\ddot{\times}\,|\iota(t)|_\beta^{p_\beta}\ \text{ whenever }\ |t|_\alpha\,\dot{\le}\,\mu$$

for all $k\in□$.

Definition 1: Let (X_α,d_α) and $(Y_\beta,d_\beta^{'})$ be non-Newtonian sequence spaces. An operator $F:X_\alpha\to Y_\beta$ is *-bounded if $F(A)$ is β -bounded for every α -bounded subset A of X_α.

Definition 2: Let (X_α,d_α) and $(Y_\beta,d_\beta^{'})$ be non-Newtonian sequence spaces. An operator $F:X_\alpha\to Y_\beta$ is *-locally bounded $x_0\in X_\alpha$ if there exist α -number $\mu\,\dot{>}\,\dot{0}$ and β -number $\eta\,\ddot{>}\,\ddot{0}$ such that $F(x)\in B_{d_\beta^{'}}\left[F(x_0),\eta\right]$ for $x\in B_{d_\alpha}\left[x_0,\mu\right]$. F is *-locally bounded if it is *-locally bounded it is *-locally bounded for every $x\in X_\alpha$.

Theorem 2: Let (X_α,d_α) and $(Y_\beta,d_\beta^{'})$ be non-Newtonian metric sequence spaces. An operator $F:X_\alpha\to Y_\beta$ is *-locally bounded if F is *-bounded.

Theorem 3: If the function $f:□\times□_\alpha\to□_\beta$ is *-locally bounded, it is satisfies the condition $(NA_2^{'})$.[17]

II. MAIN RESULTS

Theorem 4: Let $f:□\times□_\alpha\to□_\beta$. The non-Newtonian superposition operator ${}_NP_f:\ell_{p,\alpha}\to\ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition $(NA_2^{'})$.

Proof: Assume that ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ is *-locally bounded. Let $k \in \square$ and $a \in \square_\alpha$. Let $y = (y_n)$ be as follows

$$y_n = \begin{cases} a, & n = k \\ \dot{0}, & n \neq k \end{cases}$$

for all $k \in \square$. Then $y \in \ell_{p,\alpha}$. Since ${}_N P_f$ is *-locally bounded, there exist $\mu, \eta \dot{>} \dot{0}$ such that

$$\left\| {}_N P_f(x) \ddot{-} {}_N P_f(y) \right\|_{\ell_{1,\beta}} \ddot{\le} \iota(\eta) \text{ when } \left\| x \dot{-} y \right\|_{\ell_{p,\alpha}} \dot{\le} \mu .$$
(1)

Let $b \in \square_\alpha$ with $|b \dot{-} a|_\alpha \dot{\le} \mu$. Let $x = (x_n)$ be as follows

$$x_n = \begin{cases} b, & n = k \\ \dot{0}, & n \neq k \end{cases}$$

Then $x \in \ell_{p,\alpha}$. Since

$$\|x \dot{-} y\|_{\ell_{p,\alpha}} = \left({}_\alpha\sum_{n=1}^{\infty} |x_n \dot{-} y_n|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} = \left(|b \dot{-} a|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} = |b \dot{-} a|_\alpha \dot{\le} \mu$$

it is obtained that $\left\| {}_N P_f(x) \ddot{-} {}_N P_f(y) \right\|_{\ell_{1,\beta}} \ddot{\le} \iota(\eta)$ by (1). Then

$$\begin{aligned} |f(k,b) \ddot{-} f(k,a)|_\beta &= {}_\beta\sum_{n=1}^{\infty} |f(n,x_n) \ddot{-} f(n,y_n)|_\beta \\ &= \left\| {}_N P_f(x) \ddot{-} {}_N P_f(y) \right\|_{\ell_{1,\beta}} \\ &\ddot{\le} \iota(\eta). \end{aligned}$$

Thus $f(k,.)$ is *-locally bounded at $a \in \square_\alpha$. Since $a \in \square_\alpha$ is arbitrary, $f(k,.)$ is *-locally bounded. Then, $f(k,.)$ satisfies the condition $(NA_2^{'})$ by Theorem 3.

Conversely, assume that $f(k,.)$ satisfies the condition $(NA_2^{'})$. Let $z = (z_k) \in \ell_{p,\alpha}$. Since f satisfies the condition $(NA_2^{'})$ and ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$, by Theorem 1, there exist $\mu, \eta \dot{>} \dot{0}$ and $(c_k) \in \ell_{1,\beta}$ such that

$$|f(k,t)|_\beta \ddot{\le} c_k \ddot{+} \iota(\eta) \ddot{\times} |\iota(t)|_\beta^{p_\beta} \text{ when } |t|_\alpha \dot{\le} \mu$$
(2)

for each $k \in \square$. Let $\gamma = \dfrac{\mu}{\dot{2}} \alpha$ and let $x \in \ell_{p,\alpha}$ with $\|x \dot{-} z\|_{\ell_{p,\alpha}} \dot{\le} \gamma$. Since $\|z\|_{\ell_{p,\alpha}} \dot{<} \dot{+}\infty$, there exists a positive integer r such that

$$\|z_\lambda\|_{\ell_{p,\alpha}} = \left({}_\alpha\sum_{k=r}^{\infty} |z_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} \dot{\le} \gamma$$
(3)

for $\lambda \in \{r, r+1, ...\}$. Since $\|x \dot{-} z\|_{\ell_{p,\alpha}} \dot{\le} \gamma$, it is written that

$$\left({}_\alpha\sum_{k=r}^{\infty} |x_k \dot{-} z_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} \dot{\le} \left({}_\alpha\sum_{k=1}^{\infty} |x_k \dot{-} z_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} \dot{\le} \gamma . \quad (4)$$

From inequalities of (3), (4) and non-Newtonian Minkowski, we have that

$$\begin{aligned} \left({}_\alpha\sum_{k=r}^{\infty} |x_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} &= \left({}_\alpha\sum_{k=r}^{\infty} |x_k \dot{-} z_k \dot{+} z_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} \\ &\dot{\le} \left({}_\alpha\sum_{k=r}^{\infty} |x_k \dot{-} z_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} \dot{+} \left({}_\alpha\sum_{k=r}^{\infty} |z_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} \\ &\dot{\le} \gamma \dot{+} \gamma \\ &= \mu. \end{aligned}$$

Then $|x_k|_\alpha \dot{\le} \mu$ for all $k \ge r$. From (2), we get $|f(k,x_k)|_\beta \ddot{\le} c_k \ddot{+} \iota(\eta) \ddot{\times} |\iota(x_k)|_\beta^{p_\beta}$ for each $k \ge r$. Then

$$\begin{aligned} {}_\beta\sum_{k=r}^{\infty} |f(k,x_k)|_\beta &\ddot{\le} {}_\beta\sum_{k=r}^{\infty} c_k \ddot{+} \iota(\eta) \ddot{\times} {}_\beta\sum_{k=r}^{\infty} |\iota(x_k)|_\beta^{p_\beta} \\ &\ddot{\le} {}_\beta\sum_{k=r}^{\infty} |c_k|_\beta \ddot{+} \iota(\eta) \ddot{\times} {}_\beta\sum_{k=r}^{\infty} |\iota(x_k)|_\beta^{p_\beta} \\ &\ddot{\le} \|(c_k)\|_{\ell_{1,\beta}} \ddot{+} \iota(\eta) \ddot{\times} \iota(\mu)^{p_\beta}. \end{aligned}$$

Thus we get that

$$ {}_\beta\sum_{k=r}^{\infty}\left|f(k,x_k)\right|_\beta \ddot{\le}\left\|(c_k)\right\|_{\ell_{1,\beta}} \ddot{+}\iota(\eta)\ddot{\times}\iota(\mu)^{p_\beta}. $$
(5)

Let $m_k = {}^\beta\sup_{|t\dot{-}z_k|_\alpha \dot{\le}\gamma}\left|f(k,t)\right|_\beta$ for each $k\in\mathbb{N}$. Since f satisfies the condition $(NA_2^{'})$, it seen that $m_k \ddot{<} \ddot{+}\infty$ for all $k\in\mathbb{N}$. Since $\left\|x\dot{-}z\right\|_{\ell_{p,\alpha}} \dot{\le}\gamma$, it is written that $\left|x_k \dot{-} z_k\right|_\alpha \dot{\le}\gamma$ for all $k\in\mathbb{N}$. This implies that

$$ \left|f(k,x_k)\right|_\beta \ddot{\le} m_k $$
(6)

for all $k\in\mathbb{N}$. From (5) and (6), we find that

$$ \begin{aligned} \left\|{}_N P_f(x)\right\|_{\ell_{1,\beta}} &= {}_\beta\sum_{k=1}^{\infty}\left|f(k,x_k)\right|_\beta \\ &= {}_\beta\sum_{k=1}^{r-1}\left|f(k,x_k)\right|_\beta \ddot{+} {}_\beta\sum_{k=r}^{\infty}\left|f(k,x_k)\right|_\beta \\ &\ddot{\le} {}_\beta\sum_{k=1}^{r-1} m_k \ddot{+}\left\|(c_k)\right\|_{\ell_{1,\beta}} \ddot{+}\iota(\eta)\ddot{\times}\iota(\mu)^{p_\beta}. \end{aligned} $$

Thus

$$ \begin{aligned} \left\|{}_N P_f(x)\ddot{-}{}_N P_f(z)\right\|_{\ell_{1,\beta}} &\ddot{\le}\left\|{}_N P_f(x)\right\|_{\ell_{1,\beta}} \ddot{+}\left\|{}_N P_f(z)\right\|_{\ell_{1,\beta}} \\ &\ddot{\le} {}_\beta\sum_{k=1}^{r-1} m_k \ddot{+}\left\|(c_k)\right\|_{\ell_{1,\beta}} \ddot{+}\iota(\eta)\ddot{\times}\iota(\mu)^{p_\beta} \ddot{+}\left\|{}_N P_f(z)\right\|_{\ell_{1,\beta}}. \end{aligned} $$

Then we get that $\left\|{}_N P_f(x)\ddot{-}{}_N P_f(z)\right\|_{\ell_{1,\beta}} \ddot{\le}\varphi$ whenever

$$ \varphi = {}_\beta\sum_{k=1}^{r-1} m_k \ddot{+}\left\|(c_k)\right\|_{\ell_{1,\beta}} \ddot{+}\iota(\eta)\ddot{\times}\iota(\mu)^{p_\beta} \ddot{+}\left\|{}_N P_f(z)\right\|_{\ell_{1,\beta}}. $$

So, ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ is *-locally bounded at $z=(z_k)$.

Corollary 1: Let $f:\mathbb{N}\times\mathbb{C}_\alpha \to \mathbb{C}_\beta$ satisfy the condition (NA_2). If ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$, then ${}_N P_f$ is *-locally bounded.

Corollary 2: Let $f:\mathbb{N}\times\mathbb{C}_\alpha \to \mathbb{C}_\beta$. If ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ is *-bounded, then f satisfies the condition $(NA_2^{'})$.

Proposition 1: Let $f:\mathbb{N}\times\mathbb{C}_\alpha \to \mathbb{C}_\beta$ satisfy the condition $(NA_2^{'})$. If for each α-number $\mu \dot{>} \dot{0}$ there exists a β-number $\eta(\mu)\ddot{>}\ddot{0}$ such that

$$ {}_\beta\sum_{k=1}^{\infty}\left|f(k,x_k)\right|_\beta \ddot{\le}\eta(\mu) \quad \text{when} \quad \left({}_\alpha\sum_{k=1}^{\infty}\left|x_k\right|_\alpha^{p_\alpha}\right)^{(1/p)_\alpha} \dot{\le}\mu $$

for any finite α-sequence $x=(x_k)$, then there exists a β-sequence $c(\mu)=(c_k(\mu))\in\ell_{1,\beta}$ with $c_k(\mu)\ddot{\ge}\ddot{0}$ for all $k\in\mathbb{N}$ and $\left\|c(\mu)\right\|_{\ell_{1,\beta}} \ddot{\le}\eta(\mu)$ such that

$$ \left|f(k,t)\right|_\beta \ddot{\le} c_k(\mu)\ddot{+}\frac{\ddot{2}\ddot{\times}\eta(\mu)}{\iota(\mu)^{p_\beta}}\beta\ddot{\times}\left|\iota(t)\right|_\beta^{p_\beta} $$

when $\left|t\right|_\alpha \dot{\le}\mu$ for each $k\in\mathbb{N}$.

Proof: Take α-number $\mu\dot{>}\dot{0}$. By the hypothesis, there exists a β-number $\eta(\mu)\ddot{>}\ddot{0}$ such that

$$ {}_\beta\sum_{k=1}^{\infty}\left|f(k,x_k)\right|_\beta \ddot{\le}\eta(\mu) \quad \text{where} \quad \left({}_\alpha\sum_{k=1}^{\infty}\left|x_k\right|_\alpha^{p_\alpha}\right)^{(1/p)_\alpha} \dot{\le}\mu \qquad (7) $$

for any finite α-sequence $x=(x_k)$. Let

$$ g_\mu(k,t) = {}^\beta\max\left\{\ddot{0}, \left|f(k,t)\right|_\beta \ddot{-}\frac{\ddot{2}\ddot{\times}\eta(\mu)}{\iota(\mu)^{p_\beta}}\beta\ddot{\times}\left|\iota(t)\right|_\beta^{p_\beta}\right\} $$

and

$$ c_k(\mu) = {}^\beta\sup\left\{g_\mu(k,t) : \left|t\right|_\alpha \dot{\le}\mu\right\} $$

for each $k\in\mathbb{N}$. Let $k\in\mathbb{N}$ and $t\in\mathbb{C}_\alpha$ provided $\left|t\right|_\alpha \dot{\le}\mu$. By the definition of $g_\mu(k,t)$, since $\left|f(k,t)\right|_\beta \ddot{\le}\frac{\ddot{2}\ddot{\times}\eta(\mu)}{\iota(\mu)^{p_\beta}}\beta\ddot{\times}\left|\iota(t)\right|_\beta^{p_\beta}$ when $g_\mu(k,t)=\ddot{0}$, it is written that $\left|f(k,t)\right|_\beta \ddot{\le} c_k(\mu)\ddot{+}\frac{\ddot{2}\ddot{\times}\eta(\mu)}{\iota(\mu)^{p_\beta}}\beta\ddot{\times}\left|\iota(t)\right|_\beta^{p_\beta}$.

Assume that $g_\mu(k,t)\ne\ddot{0}$. Then

$$\left|f(k,t)\right|_{\beta} = g_{\mu}\left(k,t\right) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left|\iota(t)\right|_{\beta}^{p_\beta}$$
$$\ddot{\leq} c_k(\mu) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left|\iota(t)\right|_{\beta}^{p_\beta}.$$

Thus we get that $\left|f(k,t)\right|_{\beta} \ddot{\leq} c_k(\mu) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left|\iota(t)\right|_{\beta}^{p_\beta}$ whenever $\left|t\right|_{\alpha} \dot{\leq} \mu$ for each $k \in \mathbb{N}$. Since f satisfies the condition $(NA_2^{'})$, $g_{\mu}\left(k,t\right)$ is *-bounded on every α-bounded subset of α-real numbers for all $k \in \mathbb{N}$. Then $\ddot{0} \ddot{\leq} c_k\left(\mu\right) \ddot{<} \infty$ for all $k \in \mathbb{N}$. We have that for each $\varepsilon \ddot{>} \ddot{0}$, there exists an α-sequence $a = \left(a_k\right)$ with $\left|a_k\right|_{\alpha} \dot{\leq} \mu$ such that

$$c_k(\mu) \ddot{<} g_{\mu}(k, a_k) \ddot{+} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta$$
(8)

for all $k \in \mathbb{N}$. Let define $a' = \left(a_k^{'}\right)$ as follows

$$\left(a_k^{'}\right) = \begin{cases} a_k, & g_{\mu}(k,a_k) \neq \ddot{0} \\ \dot{0}, & g_{\mu}(k,a_k) = \ddot{0} \end{cases}.$$

There exists a finite sequence $\left(n_i\right)$ with $n_1 = 1 < n_2 < \ldots < n_m = n$ such that

$$\left({}_{\alpha}\sum_{k=1}^{n}\left|a_k^{'}\right|_{\alpha}^{p_\alpha}\right)^{(1/p)_\alpha} = \left({}_{\alpha}\sum_{k=n_1}^{n_2-1}\left|a_k^{'}\right|_{\alpha}^{p_\alpha} \dot{+} {}_{\alpha}\sum_{k=n_2}^{n_3-1}\left|a_k^{'}\right|_{\alpha}^{p_\alpha} \dot{+} \ldots \dot{+} {}_{\alpha}\sum_{k=n_{m-1}}^{n}\left|a_k^{'}\right|_{\alpha}^{p_\alpha}\right)^{(1/p)_\alpha}$$

with $\frac{\mu}{\dot{2}} \alpha \dot{\leq} \left({}_{\alpha}\sum_{k=n_i}^{n_{i+1}-1}\left|a_k^{'}\right|_{\alpha}^{p_\alpha}\right)^{(1/p)_\alpha} \dot{\leq} \mu$ for $i = 1,2,\ldots,m-2$ and $\dot{0} \dot{\leq} \left({}_{\alpha}\sum_{k=n_{m-1}}^{n}\left|a_k^{'}\right|_{\alpha}^{p_\alpha}\right)^{(1/p)_\alpha} \dot{\leq} \mu$ for any $n \in \mathbb{N}$. Let $b^{(i)} = a^{'}_{\lambda_{\{n_i, n_i+1, n_i+2, \ldots, n_{i+1}-1\}}}$ and $b^{(m-1)} = a^{'}_{\lambda_{\{n_{m-1}, n_{m-1}+1, n_{m-1}+2, \ldots, n_{m-1+1}=n\}}}$ for each $i = 1,2,\ldots,m-2$. then $b^{(i)}$ is a finite α-sequence for $i = 1,2,\ldots,m-2$ and $\left({}_{\alpha}\sum_{k=1}^{\infty}\left|b_k^{(i)}\right|_{\alpha}^{p_\alpha}\right)^{(1/p)_\alpha} \dot{\leq} \mu$. From (7), we get that ${}_{\beta}\sum_{k=1}^{\infty}\left|f\left(k,b_k^{(i)}\right)\right|_{\beta} \ddot{\leq} \eta\left(\mu\right)$ for all $i = 1,2,\ldots,m-2$. So, for each $i = 1,2,\ldots,m-2$

$${}_{\beta}\sum_{k=n_i}^{n_{i+1}-1}\left|f\left(k,b_k^{(i)}\right)\right|_{\beta} \ddot{\leq} \eta\left(\mu\right) \quad (9)$$

and

$${}_{\beta}\sum_{k=n_{m-1}}^{n}\left|f\left(k,b_k^{(m-1)}\right)\right|_{\beta} \ddot{\leq} \eta\left(\mu\right). \quad (10)$$

We define that $h(k,t) = \begin{cases} f(k,t), & g_{\mu}(k,t) \ddot{>} \ddot{0} \\ \ddot{0}, & g_{\mu}(k,t) = \ddot{0} \end{cases}.$

From (9) and (10),

$${}_{\beta}\sum_{k=n_i}^{n_{i+1}-1}\left|h\left(k,b_k^{(i)}\right)\right|_{\beta} \ddot{\leq} \eta\left(\mu\right) \quad (11)$$

and

$${}_{\beta}\sum_{k=n_{m-1}}^{n}\left|h\left(k,b_k^{(m-1)}\right)\right|_{\beta} \ddot{\leq} \eta\left(\mu\right) \quad (12)$$

for $i = 1,2,\ldots,m-2$. Then we get that

$$g_{\mu}\left(k,a_k\right) = \left|h(k,a_k^{'})\right|_{\beta} \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left|\iota(a_k^{'})\right|_{\beta}^{p_\beta}. \quad (13)$$

By (8), (11), (12) and (13), we find

$$
{}_{\beta}\sum_{k=1}^{n} c_k(\mu) \ddot{<} {}_{\beta}\sum_{k=1}^{n} g_\mu(k,a_k) \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta
$$

$$
= \left(\begin{array}{l} {}_{\beta}\sum_{k=n_1}^{n_2-1} g_\mu(k,a_k) \ddot{+} {}_{\beta}\sum_{k=n_2}^{n_3-1} g_\mu(k,a_k) \\ \ddot{+} \ldots \ddot{+} {}_{\beta}\sum_{k=n_{m-1}}^{n} g_\mu(k,a_k) \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta \end{array} \right)
$$

$$
= \left(\begin{array}{l} {}_{\beta}\sum_{k=n_1}^{n_2-1} \left(\left| h(k,a_k') \right|_\beta \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(a_k') \right|_\beta^{p_\beta} \right) \\ \ddot{+} {}_{\beta}\sum_{k=n_2}^{n_3-1} \left(\left| h(k,a_k') \right|_\beta \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(a_k') \right|_\beta^{p_\beta} \right) \\ \ddot{+} \ldots \ddot{+} {}_{\beta}\sum_{k=n_{m-1}}^{n} \left(\left| h(k,a_k') \right|_\beta \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(a_k') \right|_\beta^{p_\beta} \right) \\ \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta \end{array} \right)
$$

$$
= \left(\begin{array}{l} {}_{\beta}\sum_{k=n_1}^{n_2-1} \left(\left| h(k,b_k^{(1)}) \right|_\beta \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(b_k^{(1)}) \right|_\beta^{p_\beta} \right) \ddot{+} \\ {}_{\beta}\sum_{k=n_2}^{n_3-1} \left(\left| h(k,b_k^{(2)}) \right|_\beta \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(b_k^{(2)}) \right|_\beta^{p_\beta} \right) \ddot{+} \ldots \ddot{+} \\ {}_{\beta}\sum_{k=n_{m-1}}^{n} \left(\left| h(k,b_k^{(m-1)}) \right|_\beta \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(b_k^{(m-1)}) \right|_\beta^{p_\beta} \right) \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta \end{array} \right)
$$

$$
\ddot{\leq} \left(\begin{array}{l} (\ddot{m} \ddot{-} \ddot{1}) \ddot{\times} \eta(\mu) \ddot{-} \\ \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left({}_{\beta}\sum_{k=n_1}^{n_2-1} \left| \iota(b_k^{(1)}) \right|_\beta^{p_\beta} \ddot{+} {}_{\beta}\sum_{k=n_2}^{n_3-1} \left| \iota(b_k^{(2)}) \right|_\beta^{p_\beta} \ddot{+} \ldots \ddot{+} {}_{\beta}\sum_{k=n_{m-1}}^{n} \left| \iota(b_k^{(m-1)}) \right|_\beta^{p_\beta} \right) \\ \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta \end{array} \right)
$$

$$
= \left(\begin{array}{l} (\ddot{m} \ddot{-} \ddot{1}) \ddot{\times} \eta(\mu) \ddot{-} \\ \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left({}_{\beta}\sum_{k=n_1}^{n_2-1} \left| \iota(a_k') \right|_\beta^{p_\beta} \ddot{+} {}_{\beta}\sum_{k=n_2}^{n_3-1} \left| \iota(a_k') \right|_\beta^{p_\beta} \ddot{+} \ldots \ddot{+} {}_{\beta}\sum_{k=n_{m-1}}^{n} \left| \iota(a_k') \right|_\beta^{p_\beta} \right) \\ \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta \end{array} \right)
$$

$$
\ddot{\leq} (\ddot{m} \ddot{-} \ddot{1}) \ddot{\times} \eta(\mu) \ddot{-} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} (\ddot{m} \ddot{-} \ddot{2}) \ddot{\times} \frac{\iota(\mu)^{p_\beta}}{\ddot{2}} \beta \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta
$$

$$
= \eta(\mu) \ddot{+} {}_{\beta}\sum_{k=1}^{n} \frac{\varepsilon}{\ddot{2}^{k_\beta}} \beta.
$$

Then we have that

$$
{}_{\beta}\sum_{k=1}^{\infty} c_k(\mu) = {}^{\beta}\lim_{n\to\infty} \left({}_{\beta}\sum_{k=1}^{n} c_k(\mu) \right) \ddot{\leq} \eta(\mu) \ddot{+} \varepsilon .
$$

Since $\varepsilon \ddot{>} \ddot{0}$ is arbitrary, it is written that ${}_{\beta}\sum_{k=1}^{\infty} c_k(\mu) \ddot{\leq} \eta(\mu)$. Thus we get that $\left\| (c_k(\mu)) \right\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$.

Theorem 5: Let $f : \mathbb{N} \times \mathbb{C}_\alpha \to \mathbb{C}_\beta$. The non-Newtonian superposition operator ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ is *-bounded if and only if for every α-number $\mu \dot{>} \dot{0}$, there exist a β-number $\eta(\mu) \ddot{>} \ddot{0}$ and a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ such that

$$
\left| f(k,t) \right|_\beta \ddot{\leq} c_k(\mu) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(t) \right|_\beta^{p_\beta} \quad \text{when} \quad |t|_\alpha \dot{\leq} \mu
$$

for each $k \in \mathbb{N}$.

Proof: Assume that ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ is *-bounded. Let $x = (x_k)$ a finite α-sequence with $\|x\|_{\ell_{p,\alpha}} \dot{\leq} \mu$. Then $\left({}_{\alpha}\sum_{k=1}^{\infty} |x_k|_\alpha^{p_\alpha} \right)^{(1/p)_\alpha} \dot{\leq} \mu$. Since ${}_N P_f$ is *-bounded, there exists a β-number $\eta(\mu) \ddot{>} \ddot{0}$ such that $\left\| {}_N P_f(x) \right\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu) \ddot{<} \ddot{+}\infty$. The function f satisfies the condition (NA_2') by Corollary 2. Then there exists a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ with $\|c(\mu)\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$ such that

$$
\left| f(k,t) \right|_\beta \ddot{\leq} c_k(\mu) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \left| \iota(t) \right|_\beta^{p_\beta} \quad \text{when} \quad |t|_\alpha \dot{\leq} \mu
$$

for each $k \in \mathbb{N}$ by Proposition 1.

Conversely, let $\mu \dot{>} \dot{0}$ and $x \in \ell_{p,\alpha}$ with $\|x\|_{\ell_{p,\alpha}} \dot{\leq} \mu$

. Then $\left({}_{\alpha}\sum_{k=1}^{\infty} |x_k|_{\alpha}^{p_\alpha} \right)^{(1/p)_\alpha} \dot{\leq} \mu$ and $|x_k|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$. By hypothesis, there exist a β-number $\eta(\mu) \ddot{>} \ddot{0}$ and a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ such that

$$|f(k,x_k)|_\beta \ddot{\leq} c_k(\mu) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} |\iota(x_k)|_\beta^{p_\beta}$$

for each $k \in \mathbb{N}$. Since ${}_N P_f(x) = \left(f(k,x_k) \right)_{k=1}^{\infty} \in \ell_{1,\beta}$, we get that

$$\begin{aligned} \|{}_N P_f(x)\|_{\ell_{1,\beta}} &= {}_\beta\sum_{k=1}^{\infty} |f(k,x_k)|_\beta \\ &\ddot{\leq} {}_\beta\sum_{k=1}^{\infty} (c_k(\mu)) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} \iota(\mu)^{p_\beta} \\ &\ddot{\leq} {}_\beta\sum_{k=1}^{\infty} |c_k(\mu)|_\beta \ddot{+} \ddot{2} \ddot{\times} \eta(\mu) \\ &= \|(c(\mu))\|_{\ell_{1,\beta}} \ddot{+} \ddot{2} \ddot{\times} \eta(\mu). \end{aligned}$$

Thus ${}_N P_f$ is*-bounded.

Example: Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be as follows

$$f(k,t) = \left(\frac{\ddot{1}}{\ddot{7}^{k_\beta}} \beta \ddot{+} |\iota(t)|_\beta^{p_\beta} \right) \ddot{\times} |\iota(t)|_\beta$$

where $\alpha = I$, $\beta = \exp$ for all $k \in \mathbb{N}$ and $t \in \mathbb{R}_\alpha$. Then the function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}^+$ be as $f(k,t) = e^{\left(\frac{1}{7^k} + |t|^p \right)|t|}$ for $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Since $f(k,.)$ satisfies the condition

$(NA_2^{'})$, it is seen that ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ easily. Let $\mu > 0$ and $t \in \mathbb{R}$ with $|t| \leq \mu$. We get

$$\begin{aligned} |f(k,t)|_\beta &= e^{|In f(k,t)|} \\ &= e^{\left| In\left(e^{\left(\frac{1}{7^k} + |t|^p \right)|t|} \right) \right|} \\ &= e^{\left| \left(\frac{1}{7^k} + |t|^p \right)|t| \right|} \\ &= e^{\left(\frac{|t|}{7^k} + |t|^p . |t| \right)} \\ &\leq e^{\left(\frac{\mu}{7^k} + |t|^p . \mu \right)} \\ &= e^{\frac{\mu}{7^k}} . e^{|t|^p . \mu} \end{aligned}$$

for all $k \in \mathbb{N}$. Let take $c_k(\mu) = e^{\frac{\mu}{7^k}}$ for all $k \in \mathbb{N}$. Since

$$\|(c_k(\mu))\|_{\ell_{1,\beta}} = {}_\beta\sum_{k=1}^{\infty} e^{\frac{\mu}{7^k}} = e^{\sum_{k=1}^{\infty} \frac{\mu}{7^k}} = e^{\frac{\mu}{7} \cdot \frac{1}{1-\frac{1}{7}}} = e^{\frac{\mu}{6}} < \infty$$

it is obtained that $(c_k(\mu)) \in \ell_{1,\beta}$. If we choose $\eta(\mu) = e^{\frac{\mu^{p+1}}{2}}$, we get

$$|f(k,t)|_\beta \ddot{\leq} c_k(\mu) \ddot{+} \frac{\ddot{2} \ddot{\times} \eta(\mu)}{\iota(\mu)^{p_\beta}} \beta \ddot{\times} |\iota(t)|_\beta^{p_\beta}.$$

Therefore, ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ is *-bounded by Theorem 5.

III. CONCLUSIONS

In this study we proved that ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition $(NA_2^{'})$. Also we obtained that the necessary and sufficient conditions for *-boundedness of ${}_N P_f : \ell_{p,\alpha} \to \ell_{1,\beta}$.

REFERENCES

[1] Grossman, M. and Katz, R. (1972). *Non-Newtonian Calculus*, 1st ed., Lee Press, Pigeon Cove Massachussets.
[2] Dedagich F. and Zabreiko P.P., (1987).*Operator Superpositions in The Spaces l_p*, Sibirskii Matematicheskii Zhurnal, 28, 86-98,
[3] Tainchai, P. (1996). *Boundedness of Superposition Operators on Some Sequence Spaces*, Master of Sciences Dissertation, Graduate School of Chiang Mai University, 40, Thailand.
[4] Sama-ae, A. (1997). *Boundedness of Superposition Operators on The Sequence Spaces of Maddox*, Master of Sciences Dissertation, Graduate School of Chiang Mai University, 55, Thailand.
[5] Kolk, E. and Raidjoe, A. (2007). *The Continuity of Superposition Operators on Some Sequence Spaces Defined by*

Moduli, Czechoslovak Mathematical Journal, 57, 777-792.

[6] Çakmak, A. F. and Başar F. (2012). *Some New Results on Sequence Spaces with Respect to Non-Newtonian Calculus*, Journal of Inequalities and Applications, 228,1-12.

[7] Sağır, B. and Güngör, N. (2015). *Continuity of Superposition Operators on the Double Sequence Spaces* L_p, Filomat, 9 , 2017-2118.

[8] Duyar, C., Sağır, B. and Oğur, O. (2015). *Some Basic Topological Properties on Non- Newtonian Real Line*, British Journal of Mathematics and Computer Science, 9:4, 300-307.

[9] Sağır, B. and Güngör, N. (2015). *Locally Boundedness and Continuity of Superposition Operators on the Double Sequence Spaces* C_{r0}, Journal of Computational Analysis and Applications, 19(2), 365-377.

[10] Duyar, C. and Erdoğan, M. (2016). *On Non-Newtonian Real Number Series*, IOSR Journal of Mathematics, 12:6, IV, 34-48, doi: 0.9790/5728-1206043448.

[11] Güngör, N. and Sağır, B. (2017), *Continuity of Superposition Operators on Double Sequence Spaces of Maddox* L_p, Iranian Journal of Science and Technology Transactions A: Science,doi:10.1007/s40995-017-0266-1.

[12] Erdoğan, M. and Duyar, C. (2018). *Non-Newtonian Improper Integrals*, Journal of Science and Arts, 1(42), 49-74.

[13] Sağır, B. and Erdoğan, F. (2019). *On The Function Sequences and Series in the Non-Newtonian Calculus*, Journal of Science and Arts, 4(49), 915-936.

[14] Oğur, O. (2019). *Superposition operators on sequence spaces* $\ell_p(F)$ *derived by using matrix of Fibonacci numbers*,Linear and Multilinear Algebra, 68(2):1-12.

[15] Sağır, B. and Erdoğan, F. (2019). *On Characterization of Non-Newtonian Superposition Operators in Some Sequence Spaces,* Filomat, 33(9), 2601-2612.

[16] Güngör N. (2020), *Some Geometric Properties of The Non-Newtonian Sequence Spaces* $\ell_p(N)$, Mathematica Slovaca, vol. 70:3, 689-696.

[17] Erdoğan F. and Sağır, B. (2021). *On *- Boundedness and *-Local Boundedness of Non-Newtonian Superposition Operators in* $c_{0,\alpha}$ *and* c_α *to* $\ell_{1,\beta}$, Journal of Universal Mathematics, 4(2), 241-251.

[18] N.Sager and B. Sağır, (2021), *Some Inequalities in Quasi-Banach Algebra of Non-Newtonian Bicomplex Numbers*, Filomat, 35(7).