On The Non-Linear Diophantine Equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$

Sudhanshu Aggarwal[1*], Sanjay Kumar[2]

[1]*Assistant Professor, Department of Mathematics, National Post Graduate College, Barhalganj, Gorakhpur-273402, U.P., India*
[2]*Associate Professor, Department of Mathematics, M. S. College, Saharanpur-247001, U.P., India*
**Corresponding Author*

Abstract: **Diophantine equations are very useful while studying certain problems of coordinate geometry, cryptography, trigonometry and applied algebra. In the present paper, authors studied the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where m, r, ρ are whole numbers, for determining its solution in whole number. Results show that the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where m, r, ρ are whole numbers, has no solution in whole number.**

Keywords: **Non-linear Diophantine equation; Congruence; Modulo system; Numbers.**

Mathematics Subject Classification: **11D61, 11D72, 11D45.**

I. INTRODUCTION

To determine the solutions of Diophantine equations have many challenges for scholars due to absence of generalize methods. Congruences play an important role for solving some Diophantine equations [1]. Fermat's method of descent is also used for determining the solution of some Diophantine equations [1]. Aggarwal et al. [2] discussed the Diophantine equation $223^x + 241^y = z^2$ for solution. Existence of solution of Diophantine equation $181^x + 199^y = z^2$ was given by Aggarwal et al. [3]. Bhatnagar and Aggarwal [4] proved that the exponential Diophantine equation $421^p + 439^q = r^2$ has no solution in whole number. Gupta and Kumar [5] gave the solutions of exponential Diophantine equation $n^x + (n+3m)^y = z^{2k}$.

Kumar et al. [6] studied exponential Diophantine equation $601^p + 619^q = r^2$ and proved that this equation has no solution in whole number. The non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ are studied by Kumar et al. [7]. They determined that the equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ are not solvable in non-negative integers. Kumar et al. [8] examined the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They proved that the equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$ are not solvable in whole numbers. Mishra et al. [9] gave the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ and proved that the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ has no solution in whole number.

Diophantine equations help us for finding the integer solution of famous Pythagoras theorem and Pell's equation [10-11]. Sroysang [12, 15] studied the Diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$. He determined that $\{x = 1, y = 0, z = 3\}$ is the unique solution of the equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$. Sroysang [13] studied the Diophantine equation $31^x + 32^y = z^2$ and determined that it has no positive integer solution. Sroysang [14] discussed the Diophantine equation $3^x + 5^y = z^2$. Goel et al. [16] discussed the exponential Diophantine equation ${M_5}^p + {M_7}^q = r^2$ and proved that this equation has no solution in whole number.

Kumar et al. [17] proved that the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$ has no solution in whole number. The exponential Diophantine equation $(7^{2m}) + (6r+1)^n = z^2$ has studied by Kumar et al. [18]. Aggarwal and Sharma [19] studied the non-linear Diophantine equation $379^x + 397^y = z^2$ and proved that this equation has no solution in whole number. Aggarwal and others [20-22] studied the Diophantine equations $193^x + 211^y = z^2, 313^x + 331^y = z^2$ and $331^x + 349^y = z^2$. They proved that these equations have no solution in whole number.

The main object of the present paper is to determine the solution of non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where m, r, ρ are whole numbers, in whole numbers.

Preliminaries:

Lemma: 1 The non-linear Diophantine equation $[19]^{2m} + 1 = \rho^2$, *where* m, ρ *are the whole numbers, is not solvable in whole number.*

Proof: Since $[19]^{2m}$ is an odd number for all whole number m, so $[19]^{2m} + 1 = \rho^2$ is an even number for all whole number m.

$\Rightarrow \rho$ is an even number.

$\Rightarrow \rho^2 \equiv 0(mod3)$ or $\rho^2 \equiv 1(mod3)$ (1)

Now, $19 \equiv 1(mod3)$, for all whole number m.

$\Rightarrow [19]^{2m} \equiv 1(mod3)$, for all whole number m.

$\Rightarrow [19]^{2m} + 1 \equiv 2(mod3)$, for all whole number m.

$\Rightarrow \rho^2 \equiv 2(mod3)$ (2)

The result of equation (2) denies the result of equation (1).

Hence the non-linear Diophantine equation $[19]^{2m} + 1 = \rho^2$, where m, ρ are the whole numbers, is not solvable in whole number.

Lemma: 2 The non-linear Diophantine equation $1 + [2^{2r+1} - 1] = \rho^2$, *where* r, ρ *are whole numbers, is not solvable in whole number.*

Proof: Since $[2^{2r+1}]$ is an even number for all whole number r so $[2^{2r+1} - 1]$ is an odd number for all whole numbers r.

Now, $1 + [2^{2r+1} - 1] = \rho^2$ is an even number for all whole numbers r.

$\Rightarrow \rho$ is an even number

$\Rightarrow \rho^2 \equiv 0(mod3)$ or $\rho^2 \equiv 1(mod3)$ (3)

Now $[2^{2r+1} - 1] \equiv 1(mod3)$, for all whole number r.

$\Rightarrow 1 + [2^{2r+1} - 1] \equiv 2(mod3)$, for all whole numbers r.

$\Rightarrow \rho^2 \equiv 2(mod3)$ (4)

The result of equation (4) denies the result of equation (3).

Hence the non-linear Diophantine equation $1 + [2^{2r+1} - 1] = \rho^2$, where r, ρ are whole numbers, is not solvable in whole number.

II. MAIN THEOREM

The non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where m, r, ρ are whole numbers, is not solvable in whole number.

Proof: The complete proof of this theorem has four parts.

Part: 1 If $m = 0$ then the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$ becomes$1 + [2^{2r+1} - 1] = \rho^2$, which is not solvable in whole numbers according to lemma 2.

Part: 2 If $r = 0$ then the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$ becomes$[19]^{2m} + 1 = \rho^2$, which is not solvable in whole numbers according to lemma 1.

Part: 3 If m, r are natural numbers, then $[19]^{2m}, [2^{2r+1} - 1]$ are odd numbers.

$\Rightarrow [19]^{2m} + [2^{2r+1} - 1] = \rho^2$ is an even number

$\Rightarrow \rho$ is an even number

$\Rightarrow \rho^2 \equiv 0(mod3)\ or\ \rho^2 \equiv 1(mod3)$ (5)

Now $19 \equiv 1(mod3)$

$\Rightarrow [19]^{2m} \equiv 1(mod3)\ and\ [2^{2r+1}] \equiv 2(mod3)$

$\Rightarrow [19]^{2m} \equiv 1(mod3)\ and\ [2^{2r+1} - 1] \equiv 1(mod3)$

$\Rightarrow [19]^{2m} + [2^{2r+1} - 1] \equiv 2(mod3)$

$\Rightarrow \rho^2 \equiv 2(mod3)$ (6)

The result of equation (6) denies the result of equation (5).

Hence the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where m, r are positive integers and ρ is whole number, is not solvable in whole number.

Part: 4 If $m, r = 0$, then $[19]^{2m} + [2^{2r+1} - 1] = 1 + 1 = 2 = \rho^2$, which is impossible because ρ is a whole number. Hence the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where $m, r = 0$ and ρ is whole number, is not solvable in whole number.

III. CONCLUSION

In this paper, authors successfully studied the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where m, r, ρ are whole numbers, for its solution in whole numbers. They determined that the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, where m, r, ρ are whole numbers, is not solvable in whole number.

REFERENCES

[1] Stark, H.M. (1970) An introduction of number theory, Markham Pub. Co., Chicago.

[2] Aggarwal, S., Sharma, S.D. and Singhal, H. (2020) On the Diophantine equation $223^x + 241^y = z^2$, International Journal of Research and Innovation in Applied Science, 5 (8), 155-156.

[3] Aggarwal, S., Sharma, S.D. and Vyas, A. (2020) On the existence of solution of Diophantine equation $181^x + 199^y = z^2$, International Journal of Latest Technology in Engineering, Management & Applied Science, 9 (8), 85-86.

[4] Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation $421^p + 439^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 128-129.

[5] Gupta, D. and Kumar, S. (2020) On the solutions of exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, International Journal of Interdisciplinary Global Studies, 14(4), 74-77.

[6] Kumar, A., Chaudhary, L. and Aggarwal, S. (2020) On the exponential Diophantine equation $601^p + 619^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 29-30.

[7] Kumar, S., Gupta, S. and Kishan, H. (2018) On the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$, Annals of Pure and Applied Mathematics, 18(1), 91-94.

[8] Kumar, S., Gupta, D. and Kishan, H. (2018) On the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$, Annals of Pure and Applied Mathematics, 18(2), 185-188.

[9] Mishra, R., Aggarwal, S. And Kumar, A. (2020) On the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$, International Journal of Interdisciplinary Global Studies, 14(4), 78-79.

[10] Mordell, L.J. (1969) Diophantine equations, Academic Press, London, New York.

[11] Sierpinski, W. (1988) Elementary theory of numbers, 2nd edition, North-Holland, Amsterdam.

[12] Sroysang, B. (2012) More on the Diophantine equation $8^x + 19^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 601-604.

[13] Sroysang, B. (2012) On the Diophantine equation $31^x + 32^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 609-612.

[14] Sroysang, B. (2012) On the Diophantine equation $3^x + 5^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 605-608.
[15] Sroysang, B. (2014) On the Diophantine equation $8^x + 13^y = z^2$, International Journal of Pure and Applied Mathematics, 90(1), 69-72.
[16] Goel, P., Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation $M_5{}^p + M_7{}^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 170-171.
[17] Kumar, S., Bhatnagar, K., Kumar, A. and Aggarwal, S. (2020) On the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$, International Journal of Interdisciplinary Global Studies, 14(4), 183-184.
[18] Kumar, S., Bhatnagar, K., Kumar, N. and Aggarwal, S. (2020) On the exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$, International Journal of Interdisciplinary Global Studies, 14(4), 181-182.
[19] Aggarwal, S. and Sharma, N. (2020) On the non-linear Diophantine equation $379^x + 397^y = z^2$, Open Journal of Mathematical Sciences, 4(1), 397-399. DOI: 10.30538/oms2020.0129
[20] Aggarwal, S. (2020) On the existence of solution of Diophantine equation $193^x + 211^y = z^2$, Journal of Advanced Research in Applied Mathematics and Statistics, 5(3&4), 1-2.
[21] Aggarwal, S., Sharma, S.D. and Sharma, N. (2020) On the non-linear Diophantine equation $313^x + 331^y = z^2$, Journal of Advanced Research in Applied Mathematics and Statistics, 5(3&4), 3-5.
[22] Aggarwal, S., Sharma, S.D. and Chauhan, R. (2020) On the non-linear Diophantine equation $331^x + 349^y = z^2$, Journal of Advanced Research in Applied Mathematics and Statistics, 5(3&4), 6-8.