On The Exponential Diophantine Equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$

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***Abstract*: The class of Diophantine equations is classified in two categories, one is linear Diophantine equations and the other one is non-linear Diophantine equations. Both categories of Diophantine equations are widely used to represent the many puzzle problems in mathematical form. In the present paper, authors studied the exponential Diophantine equation $(\mathbf{13^{2m}}) + (\mathbf{6^{r+1}} + \mathbf{1})^{\mathbf{n}} = \boldsymbol{\rho}^{\mathbf{2}}$, where $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{r}, \boldsymbol{\rho}$ are whole numbers, for determining its solution in whole number. Results show that the exponential Diophantine equation $(\mathbf{13^{2m}}) + (\mathbf{6^{r+1}} + \mathbf{1})^{\mathbf{n}} = \boldsymbol{\rho}^{\mathbf{2}}$, where $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{r}, \boldsymbol{\rho}$ are whole numbers, has no solution in whole number.**

***Keywords*: Exponential Diophantine equation; Congruence; Modulo system; Numbers.**

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I. INTRODUCTION

Nowadays, scholars are very interested to determine the solution of different Diophantine equations because these equations have many applications in the field of coordinate geometry, cryptography, trigonometry and applied algebra. Finding the solution of Diophantine equations have many challenges for scholars due to absence of generalize methods. Aggarwal et al. [1] discussed the Diophantine equation $223^x + 241^y = z^2$ for solution. Existence of solution of Diophantine equation $181^x + 199^y = z^2$ was given by Aggarwal et al. [2]. Bhatnagar and Aggarwal [3] proved that the exponential Diophantine equation $421^p + 439^q = r^2$ has no solution in whole number.

Gupta and Kumar [4] gave the solutions of exponential Diophantine equation$[n^x + (n + 3m)^y = z^{2k}]$. Kumar et al. [5] studied exponential Diophantine equation $601^p + 619^q = r^2$ and proved that this equation has no solution in whole number. The non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ are studied by Kumar et al. [6]. They determined that the equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ are not solvable in non-negative integers. Kumar et al. [7] examined the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They proved that the equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$ are not solvable in whole numbers.

Mishra et al. [8] gave the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ and proved that the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ has no solution in whole number. Diophantine equations help us for finding the integer solution of famous Pythagoras theorem and Pell's equation [9-10]. Sroysang [11, 14] studied the Diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$. He determined that $\{x = 1, y = 0, z = 3\}$ is the unique solution of the equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$. Sroysang [12] studied the Diophantine equation $31^x + 32^y = z^2$ and determined that it has no positive integer solution. Sroysang [13] discussed the Diophantine equation $3^x + 5^y = z^2$.

Goel et al. [15] discussed the exponential Diophantine equation ${M_5}^p + {M_7}^q = r^2$ and proved that this equation has no solution in whole number. Kumar et al. [16] proved that the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$ has no solution in whole number. The exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$ has studied by Kumar et al. [17]. Aggarwal and Sharma [18] studied the non-linear Diophantine equation $379^x + 397^y = z^2$ and proved that this equation has no solution in whole number. Aggarwal and others [19-21] studied the Diophantine equations $193^x + 211^y = z^2, 313^x + 331^y = z^2$ and $331^x + 349^y = z^2$. They proved that these equations have no solution in whole number.

The main object of the present paper is to determine the solution of exponential Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$, where m, n, r, ρ are whole numbers, in whole numbers.

II. PRELIMINARIES

Lemma: 1 The exponential Diophantine equation $(13^{2m}) + 1 = \rho^2$, *where* m, ρ *are the whole numbers, is not solvable in whole number.*

Proof: Since (13^{2m}) is an odd number for all whole number m.

$\Rightarrow (13^{2m}) + 1 = \rho^2$ is an even number for all whole number m.

$\Rightarrow \rho$ is an even number.

$\Rightarrow \rho^2 \equiv 0(mod3)$ or $\rho^2 \equiv 1(mod3)$ (1)

Now, $13 \equiv 1(mod3)$, for all whole number m.

$\Rightarrow (13^{2m}) \equiv 1(mod3)$, for all whole number m.

$\Rightarrow (13^{2m}) + 1 \equiv 2(mod3)$, for all whole number m.

$\Rightarrow \rho^2 \equiv 2(mod3)$ (2)

The result of equation (2) denies the result of equation (1).

Hence the exponential Diophantine equation $(13^{2m}) + 1 = \rho^2$, where m, ρ are the whole numbers, is not solvable in whole number.

Lemma: 2 *The exponential Diophantine equation* $1 + (6^{r+1} + 1)^n = \rho^2$, *where* r, n, ρ *are whole numbers, is not solvable in whole number.*

Proof: Since $(6^{r+1} + 1)$ is an odd number for all whole number r so $(6^{r+1} + 1)^n$ is an odd number for all whole numbers r and n.

$\Rightarrow 1 + (6^{r+1} + 1)^n = \rho^2$ is an even number for all whole numbers r and n.

$\Rightarrow \rho$ is an even number

$\Rightarrow \rho^2 \equiv 0(mod3)$ or $\rho^2 \equiv 1(mod3)$ (3)

Now $(6^{r+1} + 1) \equiv 1(mod3)$, for all whole number r.

$\Rightarrow (6^{r+1} + 1)^n \equiv 1(mod3)$, for all whole numbers r and n.

$\Rightarrow 1 + (6^{r+1} + 1)^n \equiv 2(mod3)$, for all whole numbers r and n.

$\Rightarrow \rho^2 \equiv 2(mod3)$ (4)

The result of equation (4) denies the result of equation (3).

Hence the exponential Diophantine equation $1 + (6^{r+1} + 1)^n = \rho^2$, where r, n, ρ are whole numbers, is not solvable in whole number.

*Main Theorem***:** The exponential Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$, where m, n, r, ρ are whole numbers, is not solvable in whole number.

Proof: The complete proof of this theorem has four parts.

Part: 1 If $m = 0$ then the exponential Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$ becomes $1 + (6^{r+1} + 1)^n = \rho^2$, which is not solvable in whole numbers according to lemma 2.

Part: 2 If $n = 0$ then the exponential Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$ becomes $(13^{2m}) + 1 = \rho^2$, which is not solvable in whole numbers according to lemma 1.

Part: 3 If m, n are natural numbers, then $(13^{2m}), (6^{r+1} + 1)^n$ are odd numbers.

$\Rightarrow (13^{2m}) + (6^{r+1} + 1)^n = \rho^2$ is an even number

$\Rightarrow \rho$ is an even number

$\Rightarrow \rho^2 \equiv 0(mod3)\ or\ \rho^2 \equiv 1(mod3)$ (5)

Now $13 \equiv 1(mod3)$

$\Rightarrow (13^{2m}) \equiv 1(mod3)\ and\ (6^{r+1} + 1) \equiv 1(mod3)$

$$\Rightarrow \begin{bmatrix} (13^{2m}) \equiv 1(mod3) \\ and\ (6^{r+1} + 1)^n \equiv 1(mod3) \end{bmatrix}$$

$\Rightarrow (13^{2m}) + (6^{r+1} + 1)^n \equiv 2(mod3)$

$\Rightarrow \rho^2 \equiv 2(mod3)$ (6)

The result of equation (6) denies the result of equation (5).

Hence the Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$, where m, n are positive integers and r, ρ are whole numbers, is not solvable in whole number.

Part: 4 If $m, n = 0$, then $(13^{2m}) + (6^{r+1} + 1)^n = 1 + 1 = 2 = \rho^2$, which is impossible because ρ is a whole number. Hence exponential Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$, where $m, n = 0$ and r, ρ are whole numbers, is not solvable in whole number.

III. CONCLUSION

In this paper, authors successfully studied the exponential Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$, where m, n, r, ρ are whole numbers, for its solution in whole numbers. They determined that the exponential Diophantine equation $(13^{2m}) + (6^{r+1} + 1)^n = \rho^2$, where m, n, r, ρ are whole numbers, is not solvable in whole number.

REFERENCES

[1] Aggarwal, S., Sharma, S.D. and Singhal, H. (2020) On the Diophantine equation $223^x + 241^y = z^2$, International Journal of Research and Innovation in Applied Science, 5 (8), 155-156.

[2] Aggarwal, S., Sharma, S.D. and Vyas, A. (2020) On the existence of solution of Diophantine equation $181^x + 199^y = z^2$, International Journal of Latest Technology in Engineering, Management & Applied Science, 9 (8), 85-86.

[3] Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation $421^p + 439^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 128-129.

[4] Gupta, D. and Kumar, S. (2020) On the solutions of exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, International Journal of Interdisciplinary Global Studies, 14(4), 74-77.

[5] Kumar, A., Chaudhary, L. and Aggarwal, S. (2020) On the exponential Diophantine equation $601^p + 619^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 29-30.

[6] Kumar, S., Gupta, S. and Kishan, H. (2018) On the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$, Annals of Pure and Applied Mathematics, 18(1), 91-94.

[7] Kumar, S., Gupta, D. and Kishan, H. (2018) On the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$, Annals of Pure and Applied Mathematics, 18(2), 185-188.

[8] Mishra, R., Aggarwal, S. And Kumar, A. (2020) On the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$, International Journal of Interdisciplinary Global Studies, 14(4), 78-79.

[9] Mordell, L.J. (1969) Diophantine equations, Academic Press, London, New York.
[10] Sierpinski, W. (1988) Elementary theory of numbers, 2nd edition, North-Holland, Amsterdam.
[11] Sroysang, B. (2012) More on the Diophantine equation $8^x + 19^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 601-604.
[12] Sroysang, B. (2012) On the Diophantine equation $31^x + 32^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 609-612.
[13] Sroysang, B. (2012) On the Diophantine equation $3^x + 5^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 605-608.
[14] Sroysang, B. (2014) On the Diophantine equation $8^x + 13^y = z^2$, International Journal of Pure and Applied Mathematics, 90(1), 69-72.
[15] Goel, P., Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation $M_5^p + M_7^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 170-171.
[16] Kumar, S., Bhatnagar, K., Kumar, A. and Aggarwal, S. (2020) On the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$, International Journal of Interdisciplinary Global Studies, 14(4), 183-184.
[17] Kumar, S., Bhatnagar, K., Kumar, N. and Aggarwal, S. (2020) On the exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$, International Journal of Interdisciplinary Global Studies, 14(4), 181-182.
[18] Aggarwal, S. and Sharma, N. (2020) On the non-linear Diophantine equation $379^x + 397^y = z^2$, Open Journal of Mathematical Sciences, 4(1), 397-399. DOI: 10.30538/oms2020.0129
[19] Aggarwal, S. (2020) On the existence of solution of Diophantine equation $193^x + 211^y = z^2$, Journal of Advanced Research in Applied Mathematics and Statistics, 5(3&4), 1-2.
[20] Aggarwal, S., Sharma, S.D. and Sharma, N. (2020) On the non-linear Diophantine equation $313^x + 331^y = z^2$, Journal of Advanced Research in Applied Mathematics and Statistics, 5(3&4), 3-5.
[21] Aggarwal, S., Sharma, S.D. and Chauhan, R. (2020) On the non-linear Diophantine equation $331^x + 349^y = z^2$, Journal of Advanced Research in Applied Mathematics and Statistics, 5(3&4), 6-8.