Solution of Convolution Type Linear Volterra Integral Equations with Formable Transform

Nihan Güngör

Department of Mathematical Engineering, Gumushane University, Turkey

***Abstract*: The integral transforms have recently been the focus of the studies, because the integral transforms provide simple and minimal computations for solving complicated problems in engineering and science. In this study, the Formable transform is utilized to solve convolution type linear Volterra integral equations of the first kind and the second kind. Several examples are offered to illustrate the Formable transform approach for solving convolution type Volterra integral equations.**

***Keywords*: Integral transforms, Formable transform, Integral equations, Convolution type integral equation, Volterra integral equations.**

I. INTRODUCTION

In applied mathematics, engineering, and physics, integral equ ations have been used to solve a broad range of problems. Due to the association with differential equations, which have a broad variety of applications, the integral equations have started to enter engineering and other areas' problems, and as a result, their significance has grown in recent years. Terminology for integral equations can be found in $[10, 14, 15]$.

Linear Volterra integral equation of the second kind (LVIESK) is constructed as

$$v(t) = h(t) + \lambda \int_0^t k(t,z)v(z)dz,$$

where the unknown function $v(t)$ that will be determined, $k(t,z)$ is kernel of the equation and λ is a parameter. The first kind linear Volterra integral equation (LVIEFK) is given as

$$h(t) = \int_0^t k(t,z)v(z)dz$$

The integral transform methods are among the most widely utilized mathematical techniques to determine the analytical solutions of problems in engineering and sciences without large computation labor. There are numerous integral transforms to solve integral equations, the Laplace transform is the most popular of these methods. Asiru [5] revealed how to use the Sumudu transform to solve integral equations of the convolution type. Song and Kim [13] examined convolution type Volterra integral equations of second kind and Haarsa [9] obtained the solutions of convolution type linear Volterra integral equations of first kind by utilizing the Elzaki transform. Kumar et al. [12] showed how to find solutions of convolution type linear Volterra integral equations by aid of Mohand transform. Aggarwal et al. [1-4] used Kamal, Aboodh, Shehu and Mahgoub transformations to solve linear Volterra integral equations with an integral in the form of a convolution. Gnanavel et al. [6] used Tarig transform to find the solutions of convolution type linear Volterra equations. Güngör [7,8] applied Kharrat-Toma and Kashuri-Fundo transforms to solve convolution type linear Volterra integral equations.

A novel integral transform, the Formable transform, was described by Saadeh and Ghazal [11]. They proved some properties of this transform for handling both ordinary and partial differential equations. Additionally, they investigated the duality among the new transform and some existing transforms. The purpose of this research is to use the Formable integral transform to find exact solutions of convolution type linear Volterra integral of the first kind and the second kind without large computational work.

Here, we will provide some essential information regarding the Formable integral transform:

Definition 1.1. [11] If there exits a positive number M that satisfying

$$|f(t)| \leq Me^{\alpha t}, \alpha > 0, \forall t \geq 0$$

then the function $f(t)$ is said to have exponential order on every finite interval in $[0, +\infty)$.

Definition 1.2. [11] Over the set of functions

$$\mathcal{W} = \{f(t) \colon \exists N \in (0,\infty), \rho_i > 0 \text{ for } i = 1,2, |f(t)| < Ne^{\frac{t}{\rho_i}}, \text{if } t \in [0,\infty)\Big\},$$

the Formable integral transform of an exponential order function $f(t)$ is described as

$$\mathcal{R}[f(t)] = B(s,u) = s\int_0^\infty e^{-st} f(ut)dt.$$

This is equivalent to

$$\mathcal{R}[f(t)] = \frac{s}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)dt$$

where s and u are the variables of Formable transform.

The expression

$$\mathcal{R}^{-1}[B(s,u)] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{\frac{st}{u}} B(s,u)ds.$$

denotes the inverse Formable transform of a function $f(t)$.

Some functions' Formable transformations are presented below [11]:

$f(t)$	$\mathcal{R}[f(t)] = B(s,u)$
1	1
t	$\frac{u}{s}$
$\frac{t^n}{n!}$	$\frac{u^n}{s^n}$
$e^{\beta t}$	$\frac{s}{s-\beta u}$
$\frac{t^n}{n!}e^{\beta t}$	$\frac{su^n}{(s-\beta u)^{n+1}}$
$\sin(\beta t)$	$\frac{\beta su}{s^2+\beta^2u^2}$
$\cos(\beta t)$	$\frac{s^2}{s^2+\beta^2u^2}$
$\sinh(\beta t)$	$\frac{\beta su}{s^2-\beta^2u^2}$
$\cosh(\beta t)$	$\frac{s^2}{s^2-\beta^2u^2}$

Theorem 1.3. **(***Linearity property***)** [11] If $f_1(t)$ and $f_2(t)$ are two functions in $\mathcal{W}$, then $c_1f_1(t)+c_2f_2(t)\in\mathcal{W}$ where c_1 and c_2 are arbitrary constants, and

$$\mathcal{R}[c_1f_1(t)+c_2f_2(t)] = c_1\mathcal{R}[f_1(t)] + c_2\mathcal{R}[f_2(t)].$$

Theorem 1.4. (Formable transform of the derivative) [11] Let $f^{(k)}(t)\in\mathcal{W}$ for $k=0,1,2,\dots$. Then

$$\mathcal{R}\left[f^{(k)}(t)\right] = \frac{s^k}{u^k}B(s,u) - \sum_{i=0}^{k-1}\left(\frac{s}{u}\right)^{k-i} f^{(i)}(0).$$

*Theorem 1.5. (Convolution Theorem for Formable transform***)** [11] Let $f(t)$ and $g(t)$ defined in $\mathcal{W}$ having Formable integral transforms $F(s,u)$ and $G(s,u)$, respectively. Then

$$R[f(t)*g(t)] = \frac{u}{s}F(s,u)G(s,u)$$

where $f(t)*g(t) = \int_0^t f(t-z)g(z)dz$ is the convolution of function $f(t)$ and $g(t)$.

Theorem 1.6. (Formable transform of $t^nf(t), n\geq 1$) [11] If the function $f(t)$ in $\mathcal{W}$ is multiplied with the shift function t^n, then

$$R[t^nf(t)] = (-u)^n s\frac{\partial^n}{\partial s^n}\left[\frac{R[f(t)]}{s}\right].$$

II. MAIN RESULTS

In this paper, we focus on the convolution type kernel $k(t,z)$, which is expressed by the difference $(t-z)$. The convolution type LVIESK has the form

$$v(t) = h(t) + \lambda\int_0^t k(t-z)v(z)dz$$

and the convolution type LVIEFK is written as

$$h(t) = \int_0^t k(t-z)v(z)dz.$$

Theorem 2.1. The solution of convolution type LVIEFK

$$h(t) = \int_0^t k(t-z)v(z)dz \tag{1}$$

is given by

$$v(t) = R^{-1}[B(s,u)] = R^{-1}\left[\frac{s}{u}\frac{R[h(t)]}{R[k(t)]}\right]$$

where k is the kernel and $R[v(t)] = B(s,u)$.

Proof. We can write

$$\begin{aligned} R[h(t)] &= R\left[\int_0^t k(t-z)v(z)dz\right] \\ R[h(t)] &= R[k(t)*v(t)] \end{aligned}$$

by taking Formable transform to either side of VIEFK (1). By implementing convolution theorem for Formable transform, we get

$$R[h(t)] = \frac{u}{s}R[k(t)]R[v(t)]$$

$$R[v(t)] = \frac{s}{u}\frac{R[h(t)]}{R[k(t)]}. \tag{2}$$

Having applied the inverse Formable transform on either side of (2), we write

$$v(t) = R^{-1}\left[\frac{s}{u}\frac{R[h(t)]}{R[k(t)]}\right],$$

which represents the desired solution.

Theorem 2.2. The solution of convolution type LVIESK

$$v(t) = h(t) + \lambda\int_0^t k(t-z)v(z)dz \tag{3}$$

is given by

$$v(t) = R^{-1}[B(s,u)] = R^{-1}\left[\frac{R[h(t)]}{1-\lambda\frac{u}{s}R[k(t)]}\right]$$

where k is the kernel and $R[v(t)] = B(s,u)$.

Proof. Taking the Formable transform on either side of VIESK (3), we find

$$R[v(t)] = R\left[h(t) + \lambda \int_0^t k(t-z)v(z)dz\right]$$

$$R[v(t)] = R[h(t)] + \lambda R\left[\int_0^t k(t-z)v(z)dz\right]$$

$$R[v(t)] = R[h(t)] + \lambda R[k(t) * v(t)].$$

We find the following expressions

$$\begin{aligned} R[v(t)] &= R[h(t)] + \lambda \frac{u}{s} R[k(t)]R[v(t)] \\ R[v(t)] &= \frac{R[h(t)]}{1 - \lambda \frac{u}{s} R[k(t)]}. \qquad (4) \end{aligned}$$

by using convolution theorem for Formable transform. Having applied the inverse Formable transform on either side of (4), we obtain the solution as

$$v(t) = R^{-1}\left[\frac{R[h(t)]}{1 - \lambda \frac{u}{s} R[k(t)]}\right].$$

III. APPLICATIONS

This section explains how to solve convolution type linear Volterra integral equations utilizing the Formable transform via a few examples.

Example 3.1. Use the Formable transform technique to solve convolution type LVIEFK

$$\sinh t = \int_0^t e^{t-z} v(z)dz.$$

Let's taken $R[v(t)] = B(s,u)$. It is obtained that

$$R[\sinh t] = R\left[\int_0^t e^{t-z} v(z)dz\right]$$

$$\frac{su}{s^2 - u^2} = R[e^t * v(t)] \qquad (5)$$

by implementing the Formable transform. Utilizing convolution theorem for Formable transform on (5), we find

$$\frac{su}{s^2 - u^2} = \frac{u}{s} R[e^t]R[v(t)]$$

$$\frac{su}{s^2 - u^2} = \frac{u}{s}\frac{s}{s-u} B(s,u)$$

$$B(s,u) = \frac{s}{s+u}.$$

Therefore, we write

$$R[v(t)] = B(s,u) = \frac{s}{s+u}. \qquad (6)$$

Having applied the inverse Formable transform on either side of (6), we get

$$v(t) = R^{-1}\left[\frac{s}{s+u}\right] = e^{-t}.$$

Consequently, we arrive at the answer as

$$v(t) = e^{-t}.$$

Example 3. 2. Find the solution of convolution type LVIESK

$$v(t) = \cos t - \int_0^t (t-z)\cos(t-z)v(z)dtz$$

using the Formable transform method.

Let $R[v(t)] = B(s,u)$. Having applied the Formable transform

$$\begin{aligned} R[v(t)] &= R\left[\cos t - \int_0^t (t-z)\cos(t-z)v(z)dtz\right] \\ &= R[\cos t] - R\left[\int_0^t (t-z)\cos(t-z)v(z)dtz\right] \\ &= R[\cos t] - R[t\cos(t) * v(t)]. \end{aligned}$$

Now, by implementing convolution theorem for Formable transform, it is found as

$$\begin{aligned} B(s,u) &= R[\cos t] - \frac{u}{s} R[t\cos(t)]R[v(t)] \\ &= \frac{s^2}{s^2+u^2} - \frac{u}{s}\left(-u\frac{\partial R[\cos(t)]}{\partial s} + \frac{u}{s}R[\cos(t)]\right)B(s,u) \\ &= \frac{s^2}{s^2+u^2} - \frac{u}{s}\left(-u\frac{2su^2}{(s^2+u^2)^2} + \frac{u}{s}\frac{s^2}{s^2+u^2}\right)B(s,u) \\ &= \frac{s^2}{s^2+u^2} + \frac{u^4 - u^2 s^2}{(s^2+u^2)^2} B(s,u). \end{aligned}$$

Therefore, we obtain

$$R[v(t)] = B(s,u) = \frac{s^2+u^2}{s^2+3u^2}.$$

Operating inverse Formable transform, we find

$$\begin{aligned} v(t) &= R^{-1}\left[\frac{s^2+u^2}{s^2+3u^2}\right] \\ &= R^{-1}\left[\frac{1}{3} + \frac{2}{3}\frac{s^2}{s^2+3u^2}\right] \\ &= \frac{1}{3}R^{-1}[1] + \frac{2}{3}R^{-1}\left[\frac{s^2}{s^2+3u^2}\right] \\ &= \frac{1}{3} + \frac{2}{3}\cos(\sqrt{3}t). \end{aligned}$$

Therefore, we find the solution as

$$v(t) = \frac{1}{3} + \frac{2}{3}\cos(\sqrt{3}t).$$

Example 3.3. Take the initial value problem

$$\begin{cases} v''(t) - 2v'(t) - 3v(t) = 0 \\ \quad v(0) = 1, v'(0) = 2. \end{cases}$$

This is equivalent to Volterra integral equation

$$v(t) = 1 + \int_0^t (3t - 3z + 2)v(z)dz.$$

The equation may be expressed as

$$v(t) = 1 + 3\int_0^t (t-z)v(z)dz + 2\int_0^t v(z)dz. \qquad (7)$$

If we apply Formable transform on either side of (7), it is found as

$$R[v(t)] = R[1] + 3R\left[\int_0^t (t-z)v(z)dz\right] + 2R\left[\int_0^t v(z)dz\right]$$
$$= R[1] + 3R[t * v(t)] + 2R[1 * v(t)].$$

Let's taken $R[v(t)] = B(s,u)$. Considering the convolution theorem for Formable transform, we write

$$B(s,u) = 1 + 3\frac{u}{s}R[t]R[v(t)] + 2\frac{u}{s}R[1]R[v(t)]$$
$$= 1 + 3\frac{u^2}{s^2}B(s,u) + 2\frac{u}{s}B(s,u).$$

Therefore, we obtain

$$R[v(t)] = B(s,u) = \frac{s^2}{s^2 - 3u^2 - 2su}.$$

Then having applied the inverse Formable transform, we find

$$v(t) = R^{-1}\left[\frac{1}{4}\frac{s}{s+u} + \frac{3}{4}\frac{s}{s-3u}\right]$$
$$= \frac{1}{4}R^{-1}\left[\frac{s}{s+u}\right] + \frac{3}{4}R^{-1}\left[\frac{s}{s-3u}\right]$$
$$= \frac{1}{4}e^{-t} + \frac{3}{4}e^{3t}$$

Consequently, we have the solution as

$$v(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{3t}.$$

IV. CONCLUSIONS

The Formable transform is used to determine the exact solution of the Volterra integral equations of the convolution type. The presented examples illustrate that the Formable transform gives the exact solution of the convolution type integral equations as the other integral transforms, and further requiring less time and effort in computational work to obtain solution of equations.

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