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On the Exponential Diophantine Equation 29^x- $3^y = z^2$

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Abstract: Let x, y and z be non-negative integers. We prove that the exponential Diophantine equation $29^x - 3^y = z^2$ has the unique solution $(x, y, z) = (0,0,0)$.

Keywords:divisibility; exponential Diophantine equation; modular arithmetic method; Catalan's conjecture; Euler's criterion

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I.Introduction

An exponential Diophantine equation is the Diophantine equation with at least one unknown exponent variable. These problems have challenged many mathematicians to prove all solutions to an equation. In 1844, Catalan stated that a solution to the $x^m - y^n = 1$,where x, y, m, and n are integers and all greater than 1,is $m = y = 2$ and $n = x = 3$. This statement, known as Catalan's conjecture, was proved by Mihailescu [6] in 2004. From 2005to 2017, many mathematical researchers studied the Diophantine equation as $a^x + b^y = z^2$, where a and b are positive integers, and x, y and z are non-negative integers. The examples of the research appeared in ([1], [2],[5], [8], [9]).In 2018, Robago[7] studied the exponential Diophantine equation in the form $a^x - b^y = z^2$.He proved all the solutions of $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$.From 2018 to 2023, many articles involving these forms have been released. For example, Burshtein [3] stated that $6^x - 11^y = z^2$ has no solution when $2 \le x \le 16$, while Thongnak et al. [10]– [13] proved four exponential Diophantine equations $7^x - 5^y = z^2$, $7^x - 2^y = z^2$, $15^x - 13^y = z^2$, and $11^x - 17^y = z^2$.From previous work, it appears that the process for solving the Diophantine equation is interesting, and many exponential Diophantine equations still await solution. In this work, we determine all solutions of $29^x - 3^y = z^2$.

II. Preliminaries

Theorem: 1 (Catalan's conjecture [6]) Let a, b, x, *and* y *be integers. The Diophantine equation* $a^x - b^y = z^2$ *with* $\min\{a, b, x, y\} > 1$ *has the unique solution* $(a, b, x, y) = (3, 2, 2, 3)$.

Theorem: 2 (Euler's criterion [4]) Let p *be an odd prime and* $\gcd(a, p) = 1$. *Then* a *is a quadratic residue of* p *if and only if* $a^{(p-1)/2} \equiv 1 \pmod{p}$.

Lemma: 3 *If y is an odd positive integer, then* $3^y \equiv 3 \pmod{4}$.

Proof: Let y be an odd positive integer. We obtain that $(-1)^y \equiv -1 \pmod{4}$. Because $3 \equiv (-1) \pmod{4}$, we have $3^y \equiv (-1)^y \pmod{4} \equiv -1 \pmod{4} \equiv 3 \pmod{4}$. Hence, $3^y \equiv 3 \pmod{4}$.

Lemma: 4If x *is an integer, then* $x^2 \not\equiv -1 \pmod{3}$. □

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Proof: Suppose that there exists an integer x such that $x^2 \equiv -1 \pmod 3$. Thus, -1 is a quadratic residue of 3. By Theorem 2, we can write $(-1)^{\frac{3-1}{2}} \equiv 1 \pmod 3$ or $-1 \equiv 1 \pmod 3$, which clearly is impossible. ☐

Lemma: 5*If x is an integer, then* $x^2 \not\equiv 2 \pmod 5$.

Proof: Suppose x is an integer such that $x^2 \equiv 2 \pmod 5$ which means that 2 is a quadratic residue of 5. By applying Theorem 2, we can write $(2)^{\frac{5-1}{2}} = 2^2 \equiv 1 \pmod 5$ or $4 \equiv 1 \pmod 5$, which is impossible. ☐

III. Main Result

Main Theorem: The exponential Diophantine equation $29^x - 3^y = z^2$ where x, y and z are non-negative integers has the unique solution $(x, y, z) = (0,0,0)$.

Proof: Let x, y and z be non-negative integers such that

$$29^x - 3^y = z^2. \quad (1)$$

We separate into the following four cases:

Case 1: $x = y = 0$. By equation (1),we obtain $z = 0$. Thus, a trivial solution is $(x, y, z) = (0,0,0)$.

Case 2: $x = 0$ and $y > 0$. Equation (1) becomes $z^2 = 1 - 3^y$, contradicting the minimum of z.

Case 3: $x > 0$ and $y = 0$. By equation (1), we have

$$29^x - z^2 = 1. \quad (2)$$

If $x = 1$, then equation (2) becomes $28 = z^2$, which is impossible. If $x > 1$, then equation (2) implies that $z > 1$. By Theorem 1 **(**Catalan**'**s conjecture**)**, equation **(**2**)** has no solution.

Case 4: $x > 0$ and $y > 0$. We separate into two subcases.

Subcase 4.1: y is an odd positive integer. By equation (1) and Lemma3, we have $z^2 \equiv -2 \pmod 4$ or $z^2 \equiv 2 \pmod 4$, which is impossible since $z^2 \equiv 0,1 \pmod 4$.

Subcase 4.2: y is an even positive integer. Let $y = 2l, \exists l \in \mathbb{Z}^+$. Equation (1) implies that $\boldsymbol{z^2 \equiv (-1)^x (mod\, 3)}$. By Lemma 4, we can see that x is an even positive integer. Let $x = 2k, \exists k \in \mathbb{Z}^+$. By equation (1) again, we obtain $\boldsymbol{z^2 = 29^{2k} - 3^{2l}}$. Then, $\boldsymbol{z^2 \equiv 1 - (-1)^l (mod\, 5)}$. Since Lemma 5 state that $\boldsymbol{z^2 \not\equiv 2 (mod\, 5)}$,$\boldsymbol{l}$must be an even positive integer. Let $l = 2t, \exists t \in \mathbb{Z}^+$. It yields $y = 4t$, so $\boldsymbol{3^y \equiv 1 (mod\, 5)}$.Equation(1) can be written as $\boldsymbol{3^y \equiv (29^k - z) \cdot (29^k + z)}$.Thus there exist$\alpha \in \{0,1,2,\dots,y\}$ with $\alpha < y - \alpha$such that $\boldsymbol{29^k - z = 3^\alpha}$ and $\boldsymbol{29^k + z = 3^{y-\alpha}}$. These two equations yield $\boldsymbol{2 \cdot 29^k = 3^\alpha (1 + 3^{y-2\alpha})}$. Because $3 \nmid 2 \cdot 29^k$, it is easy to see that $\alpha = 0$, and we obtain

$$2 \cdot 29^k = 1 + 3^y. \quad (3)$$

From equation (3), we note that $2(-1)^k \equiv 2 \pmod 5$.Then k is an even positive integer and we have $29^k \equiv 1 \pmod 3$. From equation (3), it follows that $2 \equiv 1 \pmod 3$ which is impossible. ☐

IV. Conclusion

In this paper, we have studied the non-negative integer solution to the exponential Diophantine equation $29^x - 3^y = z^2$. Three Lemmas were established, and basic principles in Number theory were applied in the proof to determine solutions. The result indicates that the equation has only one solution, $(x, y, z) = (0,0,0)$.

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