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Solution of the Exponential Diophantine Equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$

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Abstract: Diophantine equations are so important in solving important real-world problems like network flow problems, pole placement problems, business investment problems, and data privacy problems that researchers are becoming more interested in developing new techniques for analyzing the nature and solutions of the various Diophantine equations. In the present study, authors examined the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are non-negative integers, for the non-negative integer solution of this equation. A result of the present study shows that it has no non-negative integer solution.

Key Words: Diophantine Equation; Integers; Solution; Catalan Conjecture.

I.Introduction

Diophantine equations are useful for solving many problems in algebra, trigonometry, and astronomy, regardless of whether they are linear or non-linear [1]. Aggarwal et al. [2] solved the Diophantine equation $\beta^x + (\beta + 18)^y = z^2$ by applying modular arithmetic method on it. Sroysang [3-5] examined the Diophantine equations $143^x + 145^y = z^2$, $4^x + 10^y = z^2$ and $3^x + 85^y = z^2$ in his study. Aggarwal et al. [6] thoroughly analyzed the Diophantine problem $181^x + 199^y = z^2$.

Hoque and Kalita [7] worked on the Diophantine equation $(p^q - 1)^x + p^{qy} = z^2$. According to Aggarwal and Sharma [8], the non-linear Diophantine equation $379^x + 397^y = z^2$ has no non-negative integer solutions. Aggarwal [9] investigated the Diophantine problem $193^x + 211^y = z^2$.Kumar et al. [10] investigated the Diophantine problem $601^p + 619^q = r^2$ for non-negative integer solutions. According to Kumar et al.'s research [11], the Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$ has no non-negative integer solutions. Bhatnagar and Aggarwal [12] state that there is no solution to the Diophantine equation $421^p + 439^q = r^2$.

Gupta et al. [13] investigated the non-linear exponential Diophantine equation $(x^a + 1)^m + (y^b + 1)^n = z^2$ and discovered that it lacks a non-negative integer solution. Sroysang [14-15] investigated the Diophantine equations $323^x + 325^y = z^2$ and $3^x + 45^y = z^2$ for their non-negative integer solutions. Gupta et al. [16] studied the non-linear exponential Diophantine equation $x^\alpha + (1 + my)^\beta = z^2$.After investigating the exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$, Aggarwal and Kumar [17] proved that it is not solvable in the set of non-negative integers.

Aggarwal and Upadhyaya [18] investigated the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$ and determined that it has a single non-negative integer solution. Goel et al. [19] used the arithmetic modular technique to investigate the Diophantine problem ${M_5}^p + {M_7}^q = r^2$. According to Kumar et al. [20], the Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$ is not solvable in the set of non-negative integers.According to Mishra et al.'s analysis [21], it is impossible to solve the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ in the set of non-negative integers. Aggarwal et al. [22] studied the Diophantine problem $143^x + 45^y = z^2$. They proved that there is only one solution in the set of non-negative integers.

Aggarwal et al. [23] examined the Diophantine problem $143^x + 485^y = z^2$in their study. According to recent study by Aggarwal et al. [24], the Diophantine problem $143^x + 85^y = z^2$has just one solution in the set of non-negative integers. The Diophantine equation $223^x + 241^y = z^2$ has no solution in the set of non-negative integers, according to Aggarwal et al. [25].

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Investigating the non-negative integer solution to the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are non-negative integers, is the main objective of this study.

II.Preliminaries

PROPOSITION 2.1 Catalan's Conjecture [26]: There is only one possible solution $(a, b, \mathcal{X}, \mathcal{Y}) = (3, 2, 2, 3)$ to the Diophantine equation $a^{\mathcal{X}} - b^{\mathcal{Y}} = 1$, where $a, b, \mathcal{X}$ and $\mathcal{Y}$ are integers such that $\min\{a, b, \mathcal{X}, \mathcal{Y}\} > 1$.

LEMMA 2.2 Non-negativeintegers cannot solve the Diophantine equation $10^{\mathcal{X}} + 1 = \mathcal{Z}^2$, where $\mathcal{X}, \mathcal{Z}$are non-negative integers.

PROOF: Assume that $10^{\mathcal{X}} + 1 = \mathcal{Z}^2$ for both $\mathcal{X}$ and $\mathcal{Z}$ to be non-negative integers. If $\mathcal{X} = 0$, then $\mathcal{Z}^2 = 2$, which is absurd. Then $\mathcal{X} \geq 1$. Now $\mathcal{Z}^2 = 10^{\mathcal{X}} + 1 \geq 10^1 + 1 = 11$. Thus $\mathcal{Z} \geq 4$. Let us now consider the equation $\mathcal{Z}^2 - 10^{\mathcal{X}} = 1$. According to Proposition 2.1, $\mathcal{X} = 1$. As a result, $\mathcal{Z}^2 = 11$. This is a conceptual contradiction. As a result, there is no non-negative integer solution to the Diophantine equation $10^{\mathcal{X}} + 1 = \mathcal{Z}^2$, where $\mathcal{X}, \mathcal{Z}$ are non-negative integers.

LEMMA 2.3 Non-negative integers cannot solve the Diophantine equation $400^{\mathcal{Y}} + 1 = \mathcal{Z}^2$, where $\mathcal{Y}, \mathcal{Z}$ are non-negative integers.

PROOF: Assume that $400^{\mathcal{Y}} + 1 = \mathcal{Z}^2$ for both $\mathcal{Y}$ and $\mathcal{Z}$ to be non-negative integers. If $\mathcal{Y} = 0$, then $\mathcal{Z}^2 = 2$, which is absurd. Then $\mathcal{Y} \geq 1$. Now $\mathcal{Z}^2 = 400^{\mathcal{Y}} + 1 \geq 400^1 + 1 = 401$. Thus $\mathcal{Z} \geq 21$. Let us now consider the equation $\mathcal{Z}^2 - 400^{\mathcal{Y}} = 1$. According to Proposition 2.1, $\mathcal{Y} = 1$. As a result, $\mathcal{Z}^2 = 401$. This is a conceptual contradiction. As a result, there is no non-negative integer solution to the Diophantine equation $400^{\mathcal{Y}} + 1 = \mathcal{Z}^2$, where $\mathcal{Y}, \mathcal{Z}$ are non-negative integers.

III.Main Results

THEOREM 3.1 Non-negative integers cannot solve the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are non-negative integers.

PROOF: Given any non-negative integers $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$, let $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$. We have $\mathcal{X} \geq 1, \mathcal{Y} \geq 1$ from Lemma 2.2 and 2.3. This implies that $\mathcal{Z}$ is an even number. Then $\mathcal{Z}^2 \equiv 0 \ (mod\ 3)$ or $\mathcal{Z}^2 \equiv 1 \ (mod\ 3)$. We see $10^{\mathcal{X}} \equiv 1 \ (mod\ 3)$ and $400^{\mathcal{Y}} \equiv 1 \ (mod\ 3)$. This concludes that $\mathcal{Z}^2 \equiv 2 \ (mod\ 3)$. This is a contradiction. As a result, there is no solution in non-negative integers to the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are non-negative integers.

COROLLARY 3.2 Non-negative integers cannot solve the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^4$, where $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are non-negative integers.

PROOF Given any non-negative integers $\mathcal{X}, \mathcal{Y}$, and $\mathcal{W}$, let $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^4$. Let $\mathcal{Z} = \mathcal{W}^2$. Then the equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^4$ becomes $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$. According to Theorem 3.1, non-negative integers cannot solve this equation. Hence, non-negative integers cannot solve the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^4$, where $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are non-negative integers.

COROLLARY 3.3 Assume $\mathcal{K}$ is a positive integer. Then non-negative integers cannot solve the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^{2\mathcal{K}}$, where $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are non-negative integers.

PROOF: Assume $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are non-negative integers such that $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^{2\mathcal{K}}$. Let $\mathcal{Z} = \mathcal{W}^{\mathcal{K}}$. Then the equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^{2\mathcal{K}}$ becomes$10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$. According to Theorem 3.1, non-negative integers cannot solve this equation. Hence, non-negative integers cannot solve the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^{2\mathcal{K}}$, where $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are non-negative integers.

COROLLARY 3.4 Assume$\mathcal{K}$is a positive integer. Then non-negative integers cannot solve the Diophantine equation$10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^{2\mathcal{K}+2}$, where $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are non-negative integers.

PROOF: Assume$\mathcal{X}, \mathcal{Y}, \mathcal{W}$are non-negative integers such that $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^{2\mathcal{K}+2}$. Let $\mathcal{Z} = \mathcal{W}^{\mathcal{K}+1}$. Then the equation $10^{\mathcal{X}} + \mathcal{Y} = \mathcal{W}^{2\mathcal{K}+2}$ becomes $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$. According to Theorem 3.1, non-negative integers cannot solve this equation. Hence, non-negative integers cannot solve the Diophantine equation the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{W}^{2\mathcal{K}+2}$, where $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ are non-negative integers.

IV.Conclusion

This study successfully analyzed the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$, where $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are non-negative integers. The study's findings demonstrate that there is no way to solve the Diophantine equation $10^{\mathcal{X}} + 400^{\mathcal{Y}} = \mathcal{Z}^2$ in the set of non-negative integers.

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