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On the Exponential Diophantine Equation $17^x - 11^y = z^2$

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Abstract: In this work, we study all solutions to the exponential Diophantine equation $17^x - 11^y = z^2$ where x, y and z are non-negative integers. The result indicates that there are two solutions, which are$(x, y, z) \in \{(0,0,0), (1,0,4)\}$.

Keywords: divisibility; exponential Diophantine equation; modular arithmetic method; Catalan's conjecture; Order of an integer modulo n

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I.Introduction

Let$a$and b be positive integers and consider the exponential Diophantine equation$a^x - b^y = z^2$, where $x, y$and$z$are unknown variables. This equation has been determined as the solution by many mathematicians.In 2018, the exponential Diophantine $4^x - p^y = z^2$, for $p = 2^q - 1$ where$q$ is a prime, was investigated by J.F.T. Rabago[5]. He showed that the set of all solutions is$(x, y, z) \in \{(q-1, 2^{q-1} - 1)\} \cup \{(0,0,0)\}$. In 2019, S.Thongnak et al. [9] determined all solutions to the exponential Diophantine equation$2^x - 3^y = z^2$. They proved that the equation has only three solutions. In the same year, Burshtein[1] conjectured that the equation $6^x - 11^y = z^2$, where $x, y$and$z$are positive integers, has only one solution. In 2020, Buosi et al. [3] studied the equation $p^x - 2^y = z^2$, where $p = k^2 + 2$is a prime number and $k$isa positive integers. He proved that the solutions (x, y, z) are $(0,0,0)$and $(1,1,k)$for $k \geq 3$. Although many exponential Diophantine equations were solved, for example, [6-8, 10-13], several problems remain unsolved. In this paper, we determine all solutions to the exponential Diophantine equation$17^x - 11^y = z^2$, where $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$are non-negative integers.

II. Preliminaries

Theorem: *1(Catalan's conjecture[4]) Let$a, b, x,$ andybe integers. The exponential Diophantine equation $a^x - b^y = z^2$ with $min\{a, b, x, y\} > 1$hasthe unique solution $(a, b, x, y) = (3,2,2,3)$.*

Definition:*2 [2]Let aand n are integers and $n > 1$ with $gcd(a, n) = 1$. The order of a modulo n is the smallest positive integer $k$such that $a^k \equiv 1 (mod\, n)$and is denoted by $\text{ord}_n a$.*

Theorem: *3[2] Let the integer ahave order k modulo n($k = \text{ord}_n a$).Then $a^k \equiv 1 (mod\, n)$ if and only if $\boldsymbol{k}|\boldsymbol{n}$.*

Lemma: *4 If x is a positive odd integer, then $6^x \equiv 2,6,7,8,10 (mod\, 11)$.*

Proof:Let$x$be a positive odd integer. By division algorithm, there exist $q, r \in \mathbb{Z}$ such that$x = 10q + r$, where $r = 1,3,5,7,9$. Then, we can write

$$6^x \equiv (6^{10})^q\, 6^r (mod\, 11)$$

$$\equiv 6^r (mod\, 11)$$

$$\equiv 6, 6^3, 6^5, 6^7, 6^9 (mod\, 11)$$

$\equiv 6,7,10,8,2 (mod\, 11)$.

Hence, $6^x \equiv 2,6,7,8,10 (mod\, 11)$.

Lemma: 5 *If $x \in \mathbb{Z}$, then $x^2 \equiv 0,1,3,4,5,9\ (mod\, 11)$. Proof:* Suppose $x \in \mathbb{Z}$. There exist $q, r \in \mathbb{Z}$ such that $x = 11q + r$,

where $0 \leq r < 11$. Hence

$$x^2 \equiv 0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2 (mod\ 11)$$
$$\equiv 0,1,3,4,5,9 (mod\ 11)$$

Hence, $x^2 \equiv 0,1,3,4,5,9 (mod\ 11)$.

III. Main Result

Main Theorem: The exponential Diophantine equation $\mathbf{17^x - 11^y = z^2}$**,** where $\boldsymbol{x, y}$ and $\boldsymbol{z}$ are non-negative integers, has two solutions(x, y, z),including$(\mathbf{0, 0, 0})$and $(\mathbf{1, 0, 4})$.

Proof: Let x, y, and $z \in \mathbb{Z}^+ \cup \{0\}$ such that

$\mathbf{17^x - 11^y = z^2}$. (1)

We separate into four cases as follows.

Case 1: $\boldsymbol{x = y = 0}$. We obtain $\boldsymbol{z = 0}$ from equation (1), so a solution to the equation is $(\boldsymbol{x, y, z}) = (\mathbf{0, 0, 0})$.

Case 2: $\boldsymbol{x = 0}$and$\boldsymbol{y} > 0$. Equation (1) becomes $\mathbf{z^2 = 1 - 11^y} < 0$, which is impossible.

Case 3: $\boldsymbol{x} > 0$and$\boldsymbol{y = 0}$. Equation (1) becomes

$\mathbf{17^x - z^2 = 1}$. (2)

If $\boldsymbol{x = 1}$, then from equation (2), we obtain $\mathbf{z^2 = 16}$yielding$\mathbf{z = 4}$. Hence, another solution to equation (1) is$(x, y, z) = (1,0,4)$.

If$\boldsymbol{x} > 1$, then equation (2) implies that $\boldsymbol{z} > 1$. By Theorem 1, there is no solution.

Case 4:$x > 0$and $y > 0$. We divide into two subcases as follows.

Subcase 4. 1: $x$is a positive odd integer. By Lemma4, we have$6^x \equiv 2,6,7,8,10 (mod\ 11)$. Equation (1) yields$\mathbf{z^2 \equiv 6^x (mod\ 11)}$implying$z^2 \equiv 2,6,7,8,10 (mod\ 11)$. This contradiction to Lemma 5.

Subcase 4.2: $x$is a positive even integer. Let$x = 2k, \exists k \in \mathbb{Z}^+$. Then, we write as $11^y = 17^{2k} - z^2$or$\mathbf{11^y = (17^k - z)(17^k + z)}$.Then, there exists $\alpha \in \{0,1,2, \ldots, y\}$ satisfying $17^k - z = 11^\alpha$ and$17^k + z = 11^{y-\alpha}$,where $\alpha < y - \alpha$. Thus, we obtain

$\mathbf{2 \cdot 17^k = 11^\alpha (1 + 11^{y-2\alpha})}$. (3)

Since $\mathbf{11 \nmid 2 \cdot 17^k}$, by equation (3),we conclude that $\alpha = 0$. Then, equation (3) becomes

$\mathbf{2 \cdot 17^k = 1 + 11^y}$. (4)

It yields that $2 \cdot 2^k \equiv 2 (mod\ 5)$or $2^k \equiv 1 (mod\ 5)$. From $\mathrm{ord}_5 2 = 4$ and Theorem 3, we get $4|k$ and let $k = 4l, \exists l \in \mathbb{Z}^+$. Then, equation (4) yields $2 \cdot 6^{4l} \equiv 1 (mod\ 11)$.Since$6^9 \equiv 2 (mod\ 11)$,$\mathrm{ord}_{11} 6 = 10$ and Theorem 3,we get $10|4l + 9$ yielding that there exist $m \in \mathbb{Z}$, such that $4l + 9 = 10m$ or $2(5m - 2l) = 9$. It is a contradiction. From all cases, the solutions(x, y, z) to the exponential Diophantine equation $17^x - 11^y = z^2$are $(\mathbf{0, 0, 0})$ and $(\mathbf{1, 0, 4})$.

IV. Conclusion

We have proved all solutions to the exponential Diophantine equation $17^x - 11^y = z^2$, where $\boldsymbol{x, y}$and $\boldsymbol{z}$ are non-negative integers. The two Lemmas have been given and applied to obtain all solutions. The result indicates that the set of the solution is $(x, y, z) \in \{(0,0,0), (1,0,4)\}$.

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