On Some New Fractional Quantum Integral Inequalities

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***Abstract*-In the present article, we give the integral inequalities for the Čebyšev functional in the case of two synchronous functions by making use of Saigo fractional q-integral. Main results provide extensions in several directions for Riemann-Lioulille and Kober q-integral operator.**

Key Words: Fractional integral inequalities, Saigo fractional q-integral operator, Riemann-Liouville fractional q-integral operators, Kobar fractional q-integral operators, q-Gauss hypergeomatric function, Integrable function

Mathematics Subject Classification: - 26D10, 26A33.

I. INTRODUCTION

Consider the functional

$$T(f,g):=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \quad (1)$$

where f and g are two integrable functions which are also synchronous on [a,b]
i.e. [(f(x) – f(y)) (g(x) – g(y))] ≥ 0, for any x , y € [a,b] (Čebyšev [7]).

The necessary notations and basic definitions are given below. For more details. (See. Gorenflo and Mainardi [11] and Podlubny [21] and Kilbas [13] and Malumud [15])

The fractional q-calculus is the q-extension of the ordinary fractional calculus. Q-calculus operators applied in the areas like ordinary fractional calculus, solution of the q-differentional and q-integral equation, q-transform analysis etc. Recently M.H. Abu-Risha et al [1] and Mansour [16] derived the fundamental set solutions for the homogenous linear sequential fractional q-difference equation with constant coefficient. For more details one may refer the [10, 12, 17].

Recently interest is developed in deriving certain inequalities for the fractional Calculus. In this connection one can refer to the paper [6, 8, 9, 14, 18, and 20].

Belarbi and Ahmani [4] investigated some new integral inequalities for Čebyšev functional of two synchronous function by making use of the Riemann-Liouville fractional integral operator and Saxena et al [23] investigated some new integral inequalities for Čebyšev functional of two synchronous function by making use of the Saigo fractional integral operator.

H. Ogunmez and U. M. Ozkan [19] derived new quantum integral inequalities for Chebyshev functional of two synchronous function.

This has motivated the authors derive certain new integral inequalities associated with Saigo fractional q-integral operator. By virtue of the results for Riemann-Liouville and Kober fractional q-integral operator can also be deduced from our findings.

II. FRACTIONAL Q-CALCULUS

Let $t_0 \in R$ and define (See [3, 5])

$$T_{t_0}=\left\{\ t: t=\ t_0 q^n, n \text{ a nonnegrative integer}\ \right\} U\ \{0\},\ 0<q<1\ .$$

If there is no confusion concerning t_0 , we will denote T_{t_0} by T . For a function $f:\ T\to R$, the nabla q-derivative of f is

$$\nabla_q f(t)=\frac{f(qt)-f(t)}{(q-1)\,t} \quad (2)$$

For all $t \in T \setminus \{0\}$. The q-integral of f is (see [19])

$$\int_0^t f(s)\,\nabla(s) = (1-q)\,t\,\sum_{t=0}^{\infty} q^i f\left(t q^i\right) \qquad (3)$$

Now

The q-exponential function is defined as

$$e_q(t) = \prod_{k=0}^{\infty}\left(1-q^k t\right), \qquad e_q(0) = 1 \qquad (4)$$

Define the q-Gamma function by

$$\Gamma_q(v) = \frac{1}{1-q}\int_0^1 \left(\frac{t}{1-q}\right)^{v-1} e_q(qt)\nabla t, \qquad e_q(0) = 1 \qquad (5)$$

Or

the q-Gamma function cf. Gasper and Rahman [10] , is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{\left(q^a;q\right)_\infty (1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}} \qquad \left(a \neq 0,-1,-2,\ldots\right) \qquad (6)$$

Where

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{i=0}^{n-1}\left(1-q^i\right),\quad n>0 \text{, and } (a;q)_\infty = \prod_{i=0}^{\infty}\left(1-aq^i\right).$$

And

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}.$$

A q-analogue of the familiar Riemann-Liouville fractional integral operator of a function

f (x) due to Agarwal [2] is defined as:

$$I_q^\alpha\ \{f(x)\} = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \left(\frac{qt}{x};q\right)_{\alpha-1} f(t)\,d_q t \qquad (7)$$

Where $R(\alpha) > 0$; $|q| < 1$

And

$$(a;q)_\alpha = \frac{(a;q)_\infty}{\left(aq^\alpha;q\right)_\infty}, \qquad \alpha \in R.$$

Also the basic analogue of the Kober fractional integral operator cf. Agarwal [2] is defined by

$$I_q^{\eta,\alpha}\ \{f(x)\} = \frac{x^{-\eta-1}}{\Gamma_q(\alpha)}\int_0^x \left(\frac{qt}{x};q\right)_{\alpha-1} t^n\ f(t)\,d_q t \qquad (8)$$

Where $R(\alpha) > 0$; $|q| < 1$; $\eta \in R$

A q-analogue of the Saigo fractional integral operator is defined by Purohit and Yadav [22]

$$I_q^{\alpha,\beta,\eta} f(x) = \frac{x^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} \int_0^x \left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\eta}; q^{\alpha}; q, q\right)\right] f(t)\, d_q t \tag{9}$$

Where $\alpha > 0,\ \beta \in R$ *and* $\left|\frac{t}{x}\right| < 1$ and η is any non-negative integer

Saigo fractional q-integral operator can be extensions of Riemann-Liouville and Kober fractional q-integral operators with the following functional relations:

$$I_q^{\alpha,-\alpha,\eta} f(x) = I_q^{\alpha} f(x), \tag{10}$$

$$I_q^{\alpha,0,\eta} f(x) = I_q^{\eta,\alpha} f(x) \tag{11}$$

For f (t) = 1 in equation (9)

$$I_q^{\alpha,\beta,\eta}(1) = \frac{x^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} \int_0^x \left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\eta}; q^{\alpha}; q, q\right)\right] d_q t \tag{12}$$

When $f(t) = x^{\lambda-1}$ in equation (9)

$$I_q^{\alpha,\beta,\eta}\left(x^{\lambda-1}\right) = \frac{\Gamma_q(\lambda)\,\Gamma_q(\lambda-\beta+\eta)}{\Gamma_q(\lambda-\beta)\,\Gamma_q(\lambda+\alpha+\eta)} x^{\lambda-\beta-1} \tag{13}$$

Put $\lambda=1$ in equation (13)

$$I_q^{\alpha,\beta,\eta}(1) = \frac{\Gamma_q(1)\,\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\,\Gamma_q(1+\alpha+\eta)} x^{-\beta} \tag{14}$$

III. MAIN RESULTS

A. Theorem 1. Let f and g be two synchronous functions on $[0, \infty]$, Then for all $t > 0,\ \alpha > 0$ we have

$$I_q^{\alpha,\beta,\eta}(f\,g)(x) \geq \frac{\Gamma_q(1-\beta)\,\Gamma_q(1+\alpha+\eta)}{\Gamma_q(1)\,\Gamma_q(1-\beta+\eta)} x^{\beta}\ I_q^{\alpha,\beta,\eta}(f)(x)\ I_q^{\alpha,\beta,\eta}(g)(x) \tag{15}$$

Proof:- Let f and g be two synchronous functions on $[0, \infty]$, then for all $\tau \geq 0,\ \rho \geq 0$

we have

$$[f(\tau) - f(\rho)]\ [g(\tau) - g(\rho)] \geq 0 \tag{16}$$

$$f(\tau)g(\tau) - f(\tau)g(\rho) - f(\rho)g(\tau) + f(\rho)g(\rho) \geq 0$$

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau) \tag{17}$$

Multiplying both sides of equation (17) by

$$\frac{x^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} \left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\eta}; q^{\alpha}; q, q\right)\right]$$

we obtain

$$\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\tau)g(\tau)$$

$$+\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\rho)g(\rho)$$

$$\geq \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\tau)g(\rho)$$

$$+\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\rho)g(\tau) \tag{18}$$

Integrating equation (18) over (0, t), we find that

$$\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\tau)g(\tau)\,d_q\tau$$

$$+\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\rho)g(\rho)\,d_q\tau$$

$$\geq \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\tau)g(\rho)\,d_q\tau$$

$$+\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\rho)g(\tau)\,d_q\tau$$

Using equation (9), we have

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)+\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\rho)g(\rho)\,d_q\tau$$

$$\geq g(\rho)\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] f(\tau)\,d_q\tau$$

$$+f(\rho)\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{tq}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}t}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] g(\tau)\,d_q\tau \tag{19}$$

Applying equations (9) and (12), it gives

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)+f(\rho)g(\rho)\,I_q^{\alpha,\beta,\eta}(1)\geq g(\rho)\,I_q^{\alpha,\beta,\eta}(f)(x)+g(\rho)\,I_q^{\alpha,\beta,\eta}(g)(x) \tag{20}$$

Multiplying both sides of equation (20) by

$$\frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]$$

Yields

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)\ \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]$$
$$+f(\rho)g(\rho)\ I_q^{\alpha,\beta,\eta}(1)\ \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]$$

$$\geq\ g(\rho)\ I_q^{\alpha,\beta,\eta}(f)(x)\ \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]$$

$$+f(\rho)\ I_q^{\alpha,\beta,\eta}(g)(x)\ \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right] \qquad (21)$$

If we integrate equation (21) over (0, t), we see that

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)\int_0^x \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]d_q\rho$$
$$+\ I_q^{\alpha,\beta,\eta}(1)\int_0^x \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]f(\rho)g(\rho)\ d_q\rho$$
$$\geq\ I_q^{\alpha,\beta,\eta}(f)(x)\int_0^x \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]g(\rho)\,d_q\rho$$

$$+\ I_q^{\alpha,\beta,\eta}(g)(x)\int_0^x \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right)\right]f(\rho)\ d_q\rho \qquad (22)$$

By virtue of the results (9), (12) and (14), it yields

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)\ I_q^{\alpha,\beta,\eta}(1)+I_q^{\alpha,\beta,\eta}(f\,g)(x)\ I_q^{\alpha,\beta,\eta}(1)$$
$$\geq\ I_q^{\alpha,\beta,\eta}(f\ )(x)\ I_q^{\alpha,\beta,\eta}(g)(x)+I_q^{\alpha,\beta,\eta}(f\ )(x)\ I_q^{\alpha,\beta,\eta}(g)(x)$$

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)\left[\,2\ I_q^{\alpha,\beta,\eta}(1)\right]\ \geq\ I_q^{\alpha,\beta,\eta}(f\ )(x)\left[\,2\ I_q^{\alpha,\beta,\eta}(g)(x)\right]$$

Or

$$I_q^{\alpha,\beta,\eta}(f\,g)(x) \quad \geq \frac{1}{I_q^{\alpha,\beta,\eta}(1)} \; I_q^{\alpha,\beta,\eta}(f\,)(x)\; I_q^{\alpha,\beta,\eta}(g\,)(x)$$

$$I_q^{\alpha,\beta,\eta}(f\,g)(x) \quad \geq \frac{\Gamma_q(1-\beta)\,\Gamma_q(1+\alpha+\eta)}{\Gamma_q(1)\,\Gamma_q(1-\beta+\eta)}\, x^{\beta}\; I_q^{\alpha,\beta,\eta}(f\,)(x)\; I_q^{\alpha,\beta,\eta}(g\,)(x)$$

the theorem is proved.

For $\beta = -\alpha$, Theorem 1 gives in term of Riemann-Liouville fractional q-integral operator

1). Corollary 1.1:-

$$I_q^{\alpha}(f\,g)(x) \quad \geq \frac{\Gamma_q(1+\alpha)}{\Gamma_q(1)}\, x^{-\alpha}\; I_q^{\alpha}(f\,)(x)\; I_q^{\alpha}(g\,)(x)$$

For $\beta = 0$, Theorem 1 gives in term of Kober fractional q-integral operator

2). Corollary 1.2:-

$$I_q^{\alpha,\eta}(f\,g)(x) \quad \geq \frac{\Gamma_q(1+\alpha+\eta)}{\Gamma_q(1+\eta)}\, x^{-\alpha}\; I_q^{\alpha,\eta}(f\,)(x)\; I_q^{\alpha,\eta}(g\,)(x)$$

Next we establish the following

B. Theorem 2:- Let f and g be two synchronous functions on $[0, \infty]$, Then for all $\alpha > 0$, β, η and ξ real numbers, we have

$$\frac{\Gamma_q(1)\,\Gamma_q(1-\beta+\xi)}{\Gamma_q(1-\beta)\,\Gamma_q(1+\alpha+\xi)}\, x^{-\beta}\; I_q^{\alpha,\beta,\eta}(f\,g)(x) + \frac{\Gamma_q(1)\,\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\,\Gamma_q(1+\alpha+\eta)}\, x^{-\beta}\; I_q^{\alpha,\beta,\xi}(f\,g)(x)$$

$$\geq I_q^{\alpha,\beta,\eta}(f\,)(x)\; I_q^{\alpha,\beta,\xi}(g)(x) + I_q^{\alpha,\beta,\xi}(f\,)(x)\; I_q^{\alpha,\beta,\eta}(g)(x) \qquad (23)$$

Proof: - Using similar arguments as in the proof of theorem 1, we can write

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)\; \frac{x^{-\beta-1}\,q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)} \left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\xi}; q^{\alpha}; q, q\right)\right]$$

$$+ f(\rho)\,g(\rho)\; I_q^{\alpha,\beta,\eta}(1)\; \frac{x^{-\beta-1}\,q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)} \left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\xi}; q^{\alpha}; q, q\right)\right]$$

$$\geq g(\rho)\; I_q^{\alpha,\beta,\eta}(f)(x)\; \frac{x^{-\beta-1}\,q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)} \left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\xi}; q^{\alpha}; q, q\right)\right]$$

$$+ f(\rho)\; I_q^{\alpha,\beta,\eta}(g)(x)\; \frac{x^{-\beta-1}\,q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)} \left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\xi}; q^{\alpha}; q, q\right)\right] \qquad (24)$$

Integrating equation (24) over (0, t) we find that

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)\int_0^x \frac{x^{-\beta-1}q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\xi};q^{\alpha};q,q\right)\right]d_q\rho$$

$$+\;I_q^{\alpha,\beta,\eta}(1)\int_0^x \frac{x^{-\beta-1}q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\xi};q^{\alpha};q,q\right)\right]f(\rho)\,g(\rho)\,d_q\rho$$

$$\geq\; I_q^{\alpha,\beta,\eta}(f)(x)\int_0^x \frac{x^{-\beta-1}q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\xi};q^{\alpha};q,q\right)\right]g(\rho)\,d_q\rho$$

$$+\;I_q^{\alpha,\beta,\eta}(g)(x)\int_0^x \frac{x^{-\beta-1}q^{-\xi(\alpha+\beta)}}{\Gamma_q(\alpha)}\left(\frac{\rho q}{x};q\right)_{\alpha-1} T_{q,\frac{q^{\alpha+1}\rho}{x}}\left[{}_2\phi_1\left(q^{\alpha+\beta},q^{-\xi};q^{\alpha};q,q\right)\right]f(\rho)\,d_q\rho$$

Using equation (12) and (14)

$$I_q^{\alpha,\beta,\eta}(f\,g)(x)\;I_q^{\alpha,\beta,\xi}(1)\;+\;I_q^{\alpha,\beta,\xi}(f\,g)(x)\;I_q^{\alpha,\beta,\eta}(1)$$
$$\geq\; I_q^{\alpha,\beta,\eta}(f)(x)\;I_q^{\alpha,\beta,\xi}(g)(x)+I_q^{\alpha,\beta,\xi}(f)(x)\;I_q^{\alpha,\beta,\eta}(g)(x) \qquad (25)$$

Or

$$\frac{\Gamma_q(1)\,\Gamma_q(1-\beta+\xi)}{\Gamma_q(1-\beta)\,\Gamma_q(1+\alpha+\xi)}\;x^{-\beta}\;I_q^{\alpha,\beta,\eta}(f\,g)(x)+\frac{\Gamma_q(1)\,\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\,\Gamma_q(1+\alpha+\eta)}\;x^{-\beta}\;I_q^{\alpha,\beta,\xi}(f\,g)(x)$$

$$\geq\; I_q^{\alpha,\beta,\eta}(f)(x)\;I_q^{\alpha,\beta,\xi}(g)(x)+I_q^{\alpha,\beta,\xi}(f)(x)\;I_q^{\alpha,\beta,\eta}(g)(x)$$

the theorem 2 is proved.

For $\beta = -\alpha$, Theorem 2 gives in term of Riemann-liouville fractional q-integral operator

1). Corollary 2.1:-

$$\frac{1}{\Gamma_q(1-\beta)}\;x^{\alpha}\;I_q^{\alpha}(f\,g)(x)+\frac{1}{\Gamma_q(1-\beta)}\;x^{\alpha}\;I_q^{\xi}(f\,g)(x)\geq I_q^{\alpha}(f)(x)\;I_q^{\xi}(g)(x)+I_q^{\xi}(f)(x)\;I_q^{\alpha}(g)(x)$$

For $\beta = 0$, Theorem 2 gives in term of Kober fractional q-integral operator

2). Corollary 2.2:-

$$\frac{\Gamma_q(1+\xi)}{\Gamma_q(1+\alpha+\xi)}\;I_q^{\alpha,\eta}(f\,g)(x)+\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}\;I_q^{\alpha,\xi}(f\,g)(x)$$

$$\geq\; I_q^{\alpha,\eta}(f)(x)\;I_q^{\alpha,\xi}(g)(x)+I_q^{\alpha,\xi}(f)(x)\;I_q^{\alpha,\eta}(g)(x)$$

Next, we establish

C. Theorem 3:- Let $(f_i)_{i=1,2,3.......n}$ be n positive increasing function on $[0,\infty]$. Then for any $t > 0$, $\alpha > 0$, β and η real numbers, we have

$$I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i\right](x) \geq \left[I_q^{\alpha,\beta,\eta}(1)\right]^{1-n} \prod_{i=1}^{n} I_q^{\alpha,\beta,\eta}(f_i)(x) \qquad (26)$$

Proof: - We prove this theorem by the method of mathematical induction Clearly for $n = 1$ it gives

$$I_q^{\alpha,\beta,\eta}(f_1)(x) \geq I_q^{\alpha,\beta,\eta}(f_1)(x) \quad \text{for all } t > 0, \quad \alpha > 0$$

Again for n = 2 , we have

$$I_q^{\alpha,\beta,\eta}(f_1 f_2)(x) \geq \left[I_q^{\alpha,\beta,\eta}(1)\right]^{-1} I_q^{\alpha,\beta,\eta}(f_1)(x)\ I_q^{\alpha,\beta,\eta}(f_2)(x)$$

where $t > 0$, $\alpha > 0$

By (induction hypothesis), it yields

$$I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n-1} f_i\right](x) \geq \left[I_q^{\alpha,\beta,\eta}(1)\right]^{2-n} \prod_{i=1}^{n-1} I_q^{\alpha,\beta,\eta}(f_i)(x) \qquad (27)$$

$(f_i)_{I=1,2,3.......n}$ are positive increasing function, then $\left[\prod_{i=1}^{n-1} f_i\right](x)$ is also an increasing function. Hence we can apply Theorem 2 to the function

$\prod_{i=1}^{n-1} f_i = g$, $f_n = f$, we obtain

$$I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n-1} f_i\right](x) = I_q^{\alpha,\beta,\eta}(f\,g)(x) \geq \left[I_q^{\alpha,\beta,\eta}(1)\right]^{-1} I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n-1} f_i\right](x)\ I_q^{\alpha,\beta,\eta}(f_n)(x)$$

Taking into account the equation (27)

we obtain

$$I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i\right](x) \geq \left[I_q^{\alpha,\beta,\eta}(1)\right]^{-1}\left[I_q^{\alpha,\beta,\eta}(1)\right]^{2-n}\left[\prod_{i=1}^{n-1} I_q^{\alpha,\beta,\eta}(f_i)\right](x)\ I_q^{\alpha,\beta,\eta}(f_i)(x)\ I_q^{\alpha,\beta,\eta}(f_n)(x)$$

$$I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i\right](x) \geq \left[I_q^{\alpha,\beta,\eta}(1)\right]^{1-n} \prod_{i=1}^{n} I_q^{\alpha,\beta,\eta}(f_i)(x)$$

For $\beta = -\alpha$, Theorem 3 gives in term of Riemann-Liouville fractional q-integral operator

1). Corollary 3.1:-

$$I_q^{\alpha}\left[\prod_{i=1}^{n} f_i\right](x) \geq \left[I_q^{\alpha}(1)\right]^{1-n} \prod_{i=1}^{n} I_q^{\alpha}(f_i)(x)$$

For β = 0, Theorem 3 gives in term of Kober fractional q-integral operator

2). Corollary 3.2:-

$$I_q^{\alpha,\eta}\left[\prod_{i=1}^{n} f_i\right](x) \geq \left[I_q^{\alpha,\eta}(1)\right]^{1-n} \prod_{i=1}^{n} I_q^{\alpha,\eta}(f_i)(x)$$

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