The Sum Numbers and Integral Sum Numbers of Graph $K_{n+2} \setminus E\left(2K_{1,r}\right)$

R.K.Samal[1], D.Mishra[2]

[1]*Department of Mathematics, I.T.E.R, Siksha 'O' Anusandhan University, Khandagiri,Bhubaneswar-751030, Odisha, India.*

[2]*Department of Mathematics, C.V.Raman College of Engineering and Technology, Bhubaneswar-751054, Odisha, India.*

***Abstract* -The sum graph $G^+(S)$ of finite set S containing positive integers is the graph (S,E) with $uv \in E$ if and only if $u+v \in S$. A simple graph G is said to be a sum graph if it is isomorphic to a sum graph for some set S of positive integers. The sum number $\sigma(G)$ of G is the smallest number of isolated vertices which when added to G result in a sum graph. The integral sum graph $G^+(S)$ of a finite set of integers S is the graph (S,E) with $uv \in E$ if and only if $u+v \in S$. A simple graph G is said to be an integral sum graph if it is isomorphic to an integral sum graph of some set S of integers. The integral sum number $\zeta(G)$ of G is the smallest number of isolated vertices which when added to G result in an integral sum graph. In this paper we prove that**

$$\sigma\left(K_{n+2} \setminus E\left(2K_{1,r}\right)\right) = \zeta\left(K_{n+2} \setminus E\left(2K_{1,r}\right)\right) = 2n, if \ n \geq 4, 2 \leq r \leq n-1.$$

Key words- Sum graphs, integral sum graphs, Complete graph, sum number, integral sum number, stars graphs, Working vertex, sum labeling, Graph ,Complete graph.

I. INTRODUCTION

The concept of sum and integral sum graphs were introduced by F.Harary [1990, 1994] .Let N be the set of all positive integers and Z denote all integers. The sum graph $G^+(S)$ of finite subset $S \subset N$ is the graph (S,E) with $uv \in E$ if and only if $u+v \in S$. A simple graph G is said to be a sum graph if it is isomorphic to a sum graph of some $S \subset N$. The sum number $\sigma(G)$ of G is the smallest number of isolated vertices which when added to G result in a sum graph. The integral sum graph $G^+(S)$ of a finite subset $S \subset Z$ is the graph (S,E) with $uv \in E$ if and only if $u+v \in S$. A simple graph G is said to be an integral sum graph if it is isomorphic to an integral sum graph of some $S \subset Z$ .The integral sum number $\zeta(G)$ of G is the smallest number of isolated vertices which when added to G result in an integral sum graph. It is clear that $\zeta(G) \leq \sigma(G)$ for G.

In a labeling of sum graph or integral sum graph, the vertices are said to be working whose labeling corresponds to an edge uv. As we know, it is very difficult to determine $\sigma(G)$ and $\zeta(G)$ in general. But for a special types of graphs such as stars, complete graphs and $K_n \setminus E(K_n)$ with $r < n$, their sum numbers and integral sum numbers have still been derived, and we list them and list of some observation that will be useful to obtain our main results below.

Observation1.([2001]). For $n \geq 5, \sigma\left(K_n \setminus \{e\}\right) = \zeta\left(K_n \setminus \{e\}\right) = 2n-4$.

Observation2.([2001]). For $n \geq 5 \, and \, r < n, \sigma\left(K_n \setminus E\left(K_r\right)\right) = \zeta\left(K_n \setminus E\left(K_r\right)\right)$.

Observation3.([2000]). For $r \geq 2, \sigma\left(K_{1,r}\right) = 1 > 0 = \zeta\left(K_{1,r}\right)$.

Observation4.([1996]). For $n \geq 4, \sigma\left(K_n\right) = \zeta\left(K_n\right) = 2n-3$.

Observation5.([2009]).Theorem. For $n \geq 5, 2 \leq r \leq n-1$

.Then

$$\sigma\left(K_{n+1} \setminus E\left(K_{1,r}\right)\right) = \zeta\left(K_{n+1} \setminus E\left(K_{1,r}\right)\right) = \begin{cases} 2n-2 \ if, & r=1 \\ 2n-3 \ if, & 2 \leq r \leq n-1 \\ 2n-4 \ if, & r=n \end{cases}$$

:

II. RESULTS

Let $K_{n+2} \setminus E\left(2K_{1,r}\right) = (V,E), V = A \cup B \cup \{c\} \cup \{b\}$ and $S = V \cup C$ where $A = \{a_1, a_2, ..., a_r\}$, $B = \{b_1, b_2, ..., b_{n-r}\}$, $E\left(2K_{1,r}\right) = \{cb, ca_i : i = 1,2,...r-1 \, and \, ba_i : i = 1,2,...r\}$,C is the isolated vertex set, c and b

are the center of the sub graphs $2K_{1,r}$ .It is clear that

$K_{n+1} \setminus E(K_{1,1}) = K_{n+1} \setminus \{e\}$ and

$K_{n+1} \setminus E(K_{1,n}) = K_n \bigcup K_1$.

Thus $\sigma(K_{n+1} \setminus E(K_{1,1})) = \zeta(K_{n+1} \setminus (K_{1,1})) = 2n-2$ (Hey et al. (2002) and Sharary (1996)),

$\sigma(K_{n+1} \setminus E(K_{1,n})) = \zeta(K_{n+1} \setminus E(K_{1,n})) = 2n-4$

(Sharary,(1996).

In this paper we prove that the sum numbers and the integral sum numbers of the graphs

$K_{n+2} \setminus E(2K_{1,r}) = 2n$, *for all* $n \ge 4$ *and* $2 \le r \le n-1$.

Lemma 2.1 Let $n \ge 5, 2 \le r \le n-1$.Then $0 \notin S$.

Proof. We argue by contradiction. If $0 \in \mathrm{S}$ then $0 \notin C$ (Otherwise, $0 \in \mathrm{C}$,for any vertex $\mathrm{a} \in \mathrm{V}$, $\mathrm{a} + 0 = \mathrm{a} \in \mathrm{S}$:but thus there is an edge between a vertex a and an isolated vertex, a contradiction.).Thus $0 \in \mathrm{V}$ .Furthermore $0 \in \{b_1, b_2, ..., b_{n-r}\}$ .So $(K_{n+2} \setminus E(2K_{1,r}))$ is an integral sum graph. (Otherwise, there is an isolated vertex $c_1 \in C$ .Since $0 + c_1 = c_1 \in S$ the vertex $0 \in \{b_1, b_2, ..., b_{n-r}\}$ has an edge between 0 and an isolated vertex c_1 but $c_1 \in C$. Assume that

$V(K_{n+2} \setminus E(2K_{1,r})) \setminus \{c\} \bigcup \{b\} = \{u_1, u_2, ..., u_n\}$

where $u_1 < u_2 < .... < u_n$, c and b are center of the sub graphs . Then we obtain at least numbers

$u_1 + u_2, u_1 + u_3, ..., u_1 + u_n, u_2 + u_n, ..., u_{n-1} + u_n$,

$c + u_{n-r}$ and $b + u_{n-r}$ such that

$u_1 + u_2 < u_1 + u_3 < ... < u_1 + u_n < u_2 + u_n < ... < u_{n-1} + u_n < b + u_{n-r} < c + u_{n-r} < c + u_{n-r+1}$ which belong to

$V(K_{n+2} \setminus E(2K_{1,r}))$

.But $|V(K_{n+2} \setminus E(2K_{1,r}))| = n + 2 \ne 2n$ for all $n \ge 4$ and $2 \le r \le n-1$,which is a contradiction.

Hence Lemma 2.1 holds.

Lemma 2.2 Let $n \ge 4, 2 \le r \le n-1$. If there exists a vertex $b_i \in B, (1 \le i \le n-r)$ such that $c + b_i \in C$ and $b + b_i \in C$ then $c + b_j \in C$ and $b + b_j \in C$ for any vertex $b_j \in B$.

Proof. Let $n \ge 4, 2 \le r \le n-1$ .For any vertex $b_j \in B \setminus \{b_i\}$ .Since $c + b_j \in C,$ $(c + b_i) + b_j = (c + b_j) + b_i \notin S$ .Then $c + b_j \in C \bigcup \{b_i\}$ .For $2 \le r \le n-3$, we argue by contradiction. If there exists a vertex $b_{j_1} \in B \setminus \{b_i\}$ such that $c + b_{j_1} = b_i$ then $c + b_l \in C$ for any vertex $b_l \in B \setminus \{b_l, b_{j_1}\}$ .Then $b_l + b_i = b_l + (c + b_{j_1}) = b_{j_1} + (c + b_l) \in S$ and this is a contradiction. For $r = n-2$, we assume that $B = \{b_1, b_2\}$. Then $c + b_1 \in C$. We argue by contradiction. If $c + b_2 = b_1$ then for any vertex $a_i \in A, a_i + (c + b_1) = (a_i + b_1) + c \notin S$ .Since $c = b_1 - b_2$, we have $a_i + b_1 \ne c$ (Otherwise, $a_i + b_1 = b_1 - b_2$ but $0 \notin S$).Then $a_i + b_1 \in A \bigcup C$ .On the other hand, $a_i + b_1 = a_i + (c + b_2) = (a_i + b_2) \in S$. For, $r = n-1, B = \{b_1\}$ .Similarly $b + b_j \in C$ for any vertex $b_j \in B$.

Hence Lemma 2.2 holds. □

Lemma 2.3 Let $n \ge 4$ and $2 \le r \le n-1$.For any distinct vertices $a_i, a_j \in A$, $a_i + a_j \ne c$ and $a_i + a_j \ne b$.

Proof .Let $n \ge 4$ and $2 \le r \le n-1$.We argue by contradiction. Assume that there exist two distinct vertices a_s , $a_t \in A$ with $s \ne t$ such that $a_s + a_t = c$.

(1) If $3 \le r \le n-1$. For vertex any $a_i \in A \setminus \{a_s, a_t\}, c + a_i = (a_s + a_t) + a_i = (a_s + a_i) + a_t \notin S$ then $a_s + a_i \ne c = a_s + a_t$ .Then $a_s + a_i \in C \bigcup \{a_t\}$. Similarly $a_i + a_t \in C \bigcup \{a_s\}$ .Firstly we prove that $a_s + a_i \ne a_t$ for any vertex $a_i \in A \setminus \{a_s, a_t\}$. We argue by contradiction. If there exists a vertex $a_{i_1} \in A \setminus \{a_s, a_t\}$ such that $a_s + a_{i_1} = a$ then we obtain the followings assertions. For $4 \le r \le n-1$ for any vertex $a_{l_2} \in (A \setminus \{a_s, a_t\} \bigcup \{a_{i_1}\}), (a_s + a_{i_2}) + a_{i_1} = (a_s + a_{i_1}) + a_{i_2} \in S$ but $a_s + a_{i_2} \in C$. For $r = 3, A = \{a_s, a_t, a_{i_1}\}$, If $a_s + a_{i_1} = a_t$ then $a_t + a_{i_1} \in C$ (Otherwise $2a_i = 0$,which is a contradiction).For any vertex $b_j \in B$. Since

$a_t + b_j = (a_s + a_{i_1}) + b_j = a_s + (a_{i_1} + b_j) \in S, a_{i_1} + b_j \in A \bigcup B$, for any vertex $b_j \in B$ .Since $a_s + a_{i_1} \in C$, $a_s + a_{i_1} = a_t + (a_{i_1} + b_j) = (a_{i_1} + a_t) + b_j \notin S$ .Then $b_j + a_{i_1} = a_t$ and $a_s + a_{i_1} = a_t$ .Thus $a_s = b_j$,this is a contradiction. Thus for any vertex $a_i \in A \setminus \{a_s, a_t\}$, $a_s + a_i \neq a_t$ .Then $a_s + a_i \in C$ .Similarly for any vertex $a_i \in A \setminus \{a_s, a_t\}$, $a_s + a_i \in C$. Secondly we prove that $a_s + b_j \neq a$ for any vertices $a_i \in A \setminus \{a_s\}$ and $\in B$ .We argue by contradiction. Assume that there are two vertices $a_{i'} \in A \setminus \{a_s\}$ and $b_{j'} \in B$ such that $a_s + b_{j'} \neq a_i$ and $a_s + b_{j'} \neq a_{i'}$ .For $4 \le r \le n-1$ and any vertex

$a_k \in A \setminus (\{a_{i'}\} \bigcup \{a_s, a_t\}), a_{i'} + a_k = (a_s + b_{j'}) + a_k$

$= (a_s + a_k) + b_{j'} \in S$, but $a_s + a_k \in C$. For $r = 3$ and any vertex $a_{i'} \in A \setminus \{a_s, a_t\}, (a_t + a_{i'}) + b_{j'} = (a_t + b_{j'}) + a_{i'} \notin S,$ So $a_t + b_{j'} \in C \bigcup \{a_{i'}\}$ .If $a_s + b_j \neq a_i$ then $a_s + b_{j'} \in C$ .Thus for any vertices $a_i \in A \setminus \{a_s\}$ and $b_j \in B$ $a_s + b_j \neq a_i$. Finally for any vertices $a_i \in A \setminus \{a_s\}$ and $b_j \in B$, $(a_s + b_j) + a_t \notin S$. For $r = 2$, $c = a_1 + a_2$,for any vertex $b_l \in B$,Since $c + b_l = (a_1 + a_2) + b_l \in S$ .So $a_1 + b_l \in A \bigcup B$ .Similarly $a_2 + b_l \in A \bigcup B$ .Since $c = a_1 + a_2, c + b_l \notin A.$ Then $c + b_l \in C \bigcup B$ .So $a_i + a_j \neq c$. Similarly $a_i + a_j \neq b$.

Hence Lemma 2.3 holds.

Lemma 2.4 Let $n \ge 4$ and $2 \le r \le n-2$.

- If there exists two distinct vertices $b_s, b_t \in B$ such that $b_s + b_t = c$ then for any vertex $a_i \in A, b_s + a_i \in C$ and $b_v + a_i \in C$.
- If there exists two distinct vertices $b_u, b_v \in B$ such that $b_u + b_v = b$ then for any vertex $a_i \in A, b_u + a_i \in C$ and $b_v + a_i \in C$.

Proof . Let $n \ge 4$ and $2 \le r \le n-2$, for any vertex $a_i \in A$, since $c + a_i = (b_s + b_t) + a_i = (b_s + a_i) + b_t \notin S$, $(b_s + a_i) \in C \bigcup \{b_t\}$,If there exists a vertex $a_{i_1} \in A$ such that $b_s + a_{i_1} = b_t$ then for any vertex $a_{i_2} \in A \setminus \{a_{i_1}\}$, because $2 \le r \le n-2$, there is at least one such vertex a_{i_2} , $(b_s + a_{i_2}) + a_{i_1} = (b_s + a_{i_1}) + a_{i_2} = b_t + a_{i_2} \in S$,but $b_s + a_{i_2} \in C$ .So there exists two distinct vertices $b_s, b_t \in B$ such that $b_s + b_t = c$ then for any vertex $a_i \in A, b_s + a_i \in C$ and $b_v + a_i \in C$ .Similarly there exists two distinct vertices $b_u, b_v \in B$ such that $b_u + b_v = b$ then for any vertex $a_i \in A, b_u + a_i \in C$ and $b_v + a_i \in C$.

Hence Lemma 2.4 holds.

Lemma 2.5 Let $n \ge 4$ and $2 \le r \le n-3$ .For any distinct vertices $b_i + b_j \in B, b_i + b_j \neq c$ and $b_i + b_j \neq b$.

Proof. Let $n \ge 4$ and $2 \le r \le n-3$ .We argue by contradiction. If there exists two distinct vertices $b_s, b_t \in B$ such that $b_s + b_t = c$ then for any vertex $a_i \in A,$ then by Lemma 2.4 $b_s + a_i \in C$ .Thus for any vertex $b_j \in B \bigcup \{b_s\}, (b_s + a_i) + b_j = (b_s + b_j) + a_i \notin S$ .Then $b_s + b_j \in C \bigcup \{c\} \bigcup \{b\} \bigcup \{a_i\}$ .Similarly $b_t + b_k \in C \bigcup \{c\} \bigcup \{b\} \bigcup \{a_i\}$ for any vertex $b_k \in B \setminus \{b_t\}$. On the other hand , for any vertex $b_k \in B \setminus \{b_t, b_s\}$ .Since $b_j + c =$ $(b_s + b_t) + b_j = (b_s + b_j) + b_t \in S, b_s + b_j \notin C \bigcup \{c\} \bigcup \{b\}$ .Thus $b_s + b_j = a_i$. For $2 \le r \le n-4$ obviously it is a contradiction, For $r = n-3$, Let $B = \{b_s, b_t, b_l\}$,from the above $b_s + b_l = a_i$ and $b_t + b_l = a_i$ .Then $b_s = b_t$,which is a contradiction. So $b_i + b_j \neq c$. Similarly $b_i + b_j \neq c$.

Hence Lemma 2.5 holds. □

Lemma 2.6 Let $n \ge 4$ and $2 \le r \le n-1$.

- If there exists two distinct vertices $a_s \in A$ and $b_t \in B$ such that $a_s + b_t = c$ then $a_s + a_l \in C$ and $a_l + b_t \in C$ for any vertex $a_l \in A \setminus \{a_s\}$.

- If there exists two distinct vertices $b_u \in A$ and $b_v \in B$ such that $b_u + b_v = b$ then $a_u + a_i \in C$ and $a_i + b_v \in C$ for any vertex $a_i \in A \setminus \{b_u\}$.

Proof . Let $n \geq 4$ and $2 \leq r \leq n-1$. Assumethat $a_s + b_t = c$, where $a_s \in A$ and $b_t \in B$ for any vertex $a_l \in A \setminus \{a_s\}$

.Since

$c + a_l = (a_s + b_t) + a_l = (a_s + a_l) + b_t \notin S, a_s + a_l \in C \bigcup \{b_t\}$

, for any vertex $a_l \in A \setminus \{a_s\}$ and $a_l + b_t \neq c =$

$a_s + b_t, a_l + b_t \in C \bigcup \{a_s\}$.

(1) If $3 \leq r \leq n-1$, we argue by contradiction. If there exist a vertex $a_{l_1} \in A \setminus \{a_s\}$ such that $a_s + a_{l_1} = b_t$ then for any vertex $a_{l_2} \in A \setminus (\{a_s\} \bigcup \{a_{l_1}\})$.Since $3 \leq r \leq n-1$, there exists at least such one vertex a_{l_2} and $(a_s + a_{l_2}) + a_{l_1} = (a_s + a_{l_1}) + a_{l_2} = a_{l_2} + b_t \in S$ but $a_s + a_{l_2} \in C$.Thus for any vertex $a_l \in A \setminus \{a_s\}$ and $a_s + a_l \in C$.The other part is also proved by contradiction. If there exists a vertex $a'_{l_1} \in A \setminus \{a_s\}$ such that $a'_{l_1} + b_t = a_s$

then for any vertex $a'_{l_2} \in A(\{a_s\} \bigcup \{a'_{l_1}\})$

, since $r \geq 3$, $(a_s + a'_{l_2}) = (a'_{l_1} + b_t) + a'_{l_2} =$

$a'_{l_1} + (a'_{l_2} + b_t) \in S$,but $b_t + a'_{l_2} \in C$.Thus

$a_l + b_t \in C$ for any vertex $a_l \in A \setminus \{a_s\}$.

(2) If $r = 2$. Let $A = \{a_s, a_t\}$, We argue by contradiction. Assume that $a_s + a_l = b_t$.

(2.1) For any vertex $b_l \in S \setminus \{b_t\}$,

$\{a_s + b_l, a_l + b_l, c + b_l\} \subseteq S$

.Since $b_l + b_t = (a_s + a_l) + b_l = (a_s + b_l) + a_l \in S$,

$a_l + b_l \in \{a_s\} \bigcup B$ and $a_s + b_l \in \{a_l\} \bigcup B$.If

$a_l + b_l = a_s$ then $a_s + b_l \neq a_l$.If not then $2b = 0$ (just because $a_l + b_l = a_s$).So $a_s + b_l \in B$, that is

$\{a_s + b_l : b_l \in B \bigcup \{b_t\}\}$ which is impossible.

(2.2) For any vertex

$a_l \in A \setminus \{a_s\}, c + a_l = (a_s + b_t) + a_l = (a_s + b_l) + b_t \notin S$

and $b + a_l \notin S$ .Since $b_l + a_l \neq c$ and $b_t + a_l \neq b$

, $b_t + a_l \in C \bigcup \{a_s\}$ .If $b_t + a_l = a_s$

,then for any vertex $b_i \in B \setminus \{b_t\}$,

$a_s + b_l = (a_l + b_t) + b_i \; a_l + (b_i + b_t) \in S$ .So

$(b_i + b_t) \in B$ that is $\{b_i + b_t : b_i \in B \setminus \{b_t\}\}$ which is impossible ,So there exists two distinct vertices $a_s \in A$ and $b_t \in B$ such that $a_s + b_t = c$ then $a_s + a_l \in C$ and $a_l + b_t \in C$ for any vertex $a_l \in A \setminus \{a_s\}$.Similarly there exists two distinct vertices $b_u \in A$ and $b_v \in B$ such that $b_u + b_v = b$ then $a_u + a_i \in C$ and $a_i + b_v \in C$ for any vertex $a_i \in A \setminus \{b_u\}$.

Hence Lemma 2.6 holds. □

Lemma 2.7 Let $n \geq 4$ and $2 \leq r \leq n-1$ **.**

- If there exists two distinct vertices $a_s \in A$ and $b_t \in B$ such that $a_s + b_t = c$ then $c + b_l \in C$ for any vertex $b_l \in B$.
- If there exists two distinct vertices $b_u \in A$ and $b_v \in B$ such that $b_u + b_v = b$ then $b + b_l \in C$ for any vertex $b_l \in B$.

Proof .Let $n \geq 4$ and $2 \leq r \leq n-1$.Assume that $a_s + b_t = c$, where $a_s \in A$ and $b_t \in B$ for any vertex $a_l \in A \setminus \{a_s\}$ by Lemma 2.6 $a_l + b_t \in C$ .If there exists a vertex $a \in A \bigcup B$ such that $c + b_t = a$ then for any vertex $a_l \in A \setminus \{a_s\}$

.Since

$a + a_l = (c + b_t) + a_l = c + (a_l + b_t) \in S, a_l + b_t \notin C$

but $a_l + b_t \in C$ .Thus $c + b_t \in C$ .Therefore by Lemma 2.2 $c + b_l \in C$ for any vertex $b_l \in B$ .Similarly $b + b_l \in C$ for any vertex $b_l \in B$.

Hence Lemma 2.7 holds. □

Lemma 2.8 Let $n \geq 4$ and $2 \leq r \leq n-2$.If there exists two vertices $a_s \in A$ and $b_t \in B$ such that, $a_s + b_t \in C$ then $a_s + b_l \in C$ for any vertex $b_l \in B$.

Proof. Let $n \geq 4$ and $2 \leq r \leq n-2$.For any vertex $b_l \in B \setminus \{b_t\}$ .Since $a_s + b_t \in C$

$(a_s + b_t) + b_l = (a_s + b_l) + b_t \notin S$, Then .Thus

$a_s + b_l \in C \bigcup \{b_t\}$ .If there exists a vertex $b_{l_1} \in B \setminus \{b_t\}$ such that $a_s + b_{l_1} = b_t$, then we can consider

the assertions as follows. For $1 \le r \le n-3, a_s + b_{l_2} \in C$ any vertex $b_{l_2} \in B \setminus \{b_t, b_{l_1}\}$. So $b_{l_1} + b_t = b_{l_2} + (a_s + b_{l_1}) = b_{l_1} + (a_s + b_{l_2}) \in S$ but $a_s + b_{l_2} \in C$. For $r = n-2$, $B = \{b_{l_1}, b_t\}$,Firstly we show $a_s + a_k \in C$ for any vertex $a_k \in A \setminus \{a_s\}$ .In fact, for any vertex $a_k \in A \setminus \{a_s\}$,
$(a_s + b_t) + a_k = (a_s + a_k) + b_t \notin S$ then $a_s + a_k \in \{b_t\} \bigcup C$ .So there exists at most one vertex a_k such that $a_s + a_k = b_t$ and denoted by a_{k_1} and $a_s + a_{k_2} \in C$ then for any vertex $a_{k_2} \in A \setminus (\{a_{k_1}\} \bigcup \{a_s\})$ .So $(a_s + a_{k_2}) + b_{l_1} = (a_s + b_{l_1}) + a_{k_2} = b_t + a_{k_2} \in S$, but $a_s + a_{k_2} \in C$ .Secondly for any vertex $a_k \in A \setminus \{a_s\}$ .Since $a_s + b_{l_1} = b_t, (a_s + a_k) + b_{l_1} = (a_s + b_{l_1}) + a_k = b_t + a_k \in S$, but $a_s + a_{k_2} \in C$ .Thus $a_s + b_l \in C$ for any vertex $b_l \in B$.

Hence Lemma 2.8 holds. □

Lemma 2.9 Let $n \ge 4$ and $3 \le r \le n-1$. If c and b are not working vertices, then $\zeta(K_{n+2} \setminus E(2K_{1,r})) \ge 2n$

Proof. Let $n \ge 4$ and $3 \le r \le n-1$ .Without loss of generality, we may assume that
$V(K_{n+2} \setminus E(2K_{1,r})) \setminus \{c\} \bigcup \{b\} = \{u_1, u_2, ..., u_n\}$
where $u_1 < u_2 < .... < u_n$ and $u_n > 0, u_{n-1} > 0, u_{n-2} > 0$ .Since c and b are not working vertices ,the inequalities $u_n + u_{n-1} > u_n$ and $u_n + u_{n-2} > u_n$ .Show that $u_n + u_{n-1} \in C$ and $u_n + u_{n-2} \in C$ .Firstly we show that $u_n + u_i \in C$ for all $i \in \{1, 2, ..., n-3\}$ .We argue by contradiction ,if not then there exist $i \le n-3$ and $j \le n$ such that $u_n + u_i = u_j$ for $j < n-1$. Since $u_n + u_i \in S$ and $(u_n + u_{n-2}) + u_i = u_j + u_{n-2} \in S$ but $u_n + u_{n-2} \in C$. Since $u_n + u_i \in S$ and $u_n + u_{n-1} \in C, u_j + u_{n-1} \in S$ but $u_n + u_{n-1} \in C$ .So $u_n + u_i \in C$ for all $i \in \{1, 2, ...n-3\}$ .Secondlywe will prove that $u_1 + u_i \in C$ for any $i \in \{1, 2, ...n-1\}$ .We argue by contradiction, if not then there exist $2 \le i \le n-1$ and $2 \le j \le n$ and $i \ne j$ such that $u_1 + u_i = u_j$ .If $n \ne j$ then $u_n + u_j \in S$ but $u_n + u_i \in C$. If $j = n$ then for any vertex $u_l \in (A \bigcup B) \setminus \{u_1, u_n\}, u_n + u_l \in S$ .Thus $\{u_1 + u_l : l \notin \{1, n\}\} \subseteq (A \bigcup B) \setminus \{u_1\}$ .Since $u_1 < u_2 < ... < u_n$ and $u_1 + u_i = u_n, u_1 + u_{n-1} = u_n$ and $u_1 + u_k = u_{k+1}$ for any $2 \le k \le n-1$. So $u_n + u_{n-1} \in S$ but $u_n + u_1 \in C$ .Hence $u_1 + u_i \in C$ for any $i \in \{2, 3, ...n-1\}$ .Thus we have obtained $2n$ numbers in C including $c + u_{n-r}, c + u_{n-r+1}$ and $b + u_{n-r}$ .Since
$u_1 + u_2 < u_1 + u_3 < .... < u_1 + u_n < u_2 + u_n < u_{n-1} + u_n < b + u_{n-r} < c + u_{n-r} < c + u_{n-r+1}$,these are $2n$ distinct numbers .Then $\zeta(K_{n+2} \setminus E(2K_{1,r})) \ge 2n$.

Hence Lemma 2.9holds.□

Lemma 2.10 Let $n \ge 4, n-2 \le r \le n-1.$ If c and b are working vertices, then $\zeta(K_{n+2} \setminus E(2K_{1,r})) \ge 2n$.

Proof. Let $n \ge 4$ *and* $n-2 \le r \le n-1$ .Assume that
$V(K_{n+2} \setminus E(2K_{1,r})) \setminus \{c\} \bigcup \{b\} = \{u_1, u_2, ..., u_n\}$
and $C_0 = C_1 \bigcup C_n$,where c and b are the centre of the sub graphs $2K_{1,r}, u_1 < u_2 < .... < u_n$,
$C_1 = \{u_1 + u_i : i = 2, 3, ..., n\}$ and
$C_n = \{u_n + u_j : j = 1, 2, ..., n-1\}$. As c and b are working vertices ,so c and b are elements of the set C_0 .Thus we have obtained at least $2n-2$ isolated vertices which are all members of $C_0 \setminus \{b\} \bigcup \{c\}$ .Thus
$\zeta(K_{n+2} \setminus E(2K_{1,r})) \ge 2n-2$
and
$u_1 + u_2 < u_1 + u_3 < ... < u_1 + u_n < u_2 + u_n < ... < u_{n-1} + u_n < b + u_{n-r} < c + u_{n-r} < c + u_{n-r+1}$ .We only need to prove that $\zeta(K_{n+2} \setminus E(2K_{1,r})) \ne 2n-2$ .If not then $\zeta(K_{n+2} \setminus E(2K_{1,r})) = 2n-2$ .Then

$C=\{u_1+u_2,u_1+u_3,....u_1+u_n,u_2+u_n,...,u_{n-1}+u_n,b+u_{n-r},c+u_{n-r},c+u_{n-r+1}\}\setminus\{b\}\cup\{c\}$

.Assume that $c=u_{t_k}+u_i\in C_0$ where $k\in\{1,2\}$ and $\{t_1,t_2\}=\{1,n\}$. Then $c+u_h\in C_0$ where u_h is any neighbor of c .Similarly $b=u_{p_k}+u_i\in C$ where $k\in\{1,2\}$ and $\{p_1,p_2\}=\{1,n\}$. Then $b+u_h\in C_0$ where u_h is any neighbor of b.

Case-1. $r=n-1.$

Assume that $c=u_{t_1}+u_i$ where $1\le i\le n$ and $i\neq t_1$ .By Lemma 2.3.implies one of $\{u_{t_1},u_i\}\in B$ and the other is in A .By the structure of the graph $K_{n+2}\setminus E(2K_{1,r})$,the vertex c have to adjacent to the unique vertex of B ,which is in the set $\{u_{t_1},u_i\}$ .Similarly $b=u_{p_1}+u_i$ where $1\le i\le n$ and $i\neq p_1$. By Lemma 2.3.implies one of $\{u_{p_1},u_i\}\in B$ and the other is in A. By the structure of the graph $K_{n+2}\setminus E(2K_{1,r})$,the vertex b have to adjacent to the unique vertex of B which is in the set $\{u_{p_1},u_i\}$.

Case-2. $r=n-2.$

In this case ,we assume that $c=u_{t_1}+u_i$ where $1\le i\le n$ and $i\neq t_1$. By Lemma 2.3 u_{t_1} and u_i are adjacent to c. Then $c+u_{t_1}=u_{t_2}+u_j$ where $1\le j\le n$ and $j\notin\{t_1,t_2\}$ .Similarly $b=u_p+u_i$ where $1\le i\le n$ and $i\neq p_1$. By Lemma 2.3 u_{p_1} and u_i are adjacent to b .Then $b+u_{p_1}=u_{p_2}+u_j$ where $1\le j\le n$ and $j\notin\{p_1,p_2\}$ .Hence Lemma 2.10 holds. □

Lemma 2.11 For $n\ge 4, 2\le r\le n-1, \sigma\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)\le 2n$.

Proof . Let $V=V\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)=\{u_1,u_2,....u_n\}\cup\{c\}\cup\{b\}$ and $C=\{c_k:k=1,2,....2n\}$, $S=V\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)\cup\{c\}\cup\{b\}$, b and c are the centre of the sub graphs $2K_{1,r}$.

Firstly, we consider the labeling of $\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)\cup(2n)K_1$ as follows:

$u_i=(i-1)\times 10+1,\ i=1,2,....n;$

$c=(n+r)\times 10+1;$

$b=(n+r-1)\times 10+1;$

$c_k=k\times 10+2\ ,\ k=1,2,....2n.$

Secondly, let us verify that this is the sum labeling in detail.

(1) The vertices of S are distinct.

(2) For any vertices $u_i\in\{u_1,u_2,...u_n\}$ and $c_k\in C$,since $u_i+c_k\equiv 3(\mathrm{mod}10), u_i+c_k\notin S$.

(3) For any distinct vertices $c_k,c_i\in C$,since $c_i+c_k\equiv 4(\mathrm{mod}10), c_i+c_k\notin S.$

(4) For any vertices $c_k\in C$,since $c+c_k\equiv 3(\mathrm{mod}10), c+c_k\notin S.$

(5) For any vertices $c_k\in C$,since $b+c_k\equiv 3(\mathrm{mod}10), b+c_k\notin S.$

(6) Let $1\le i\neq j\le n.$ For any vertices $u_i,u_j\in\{u_1,u_2,...u_n\}$, $u_i+u_j=(i+j-2)\times 10+2=c_{i+j-2}.$

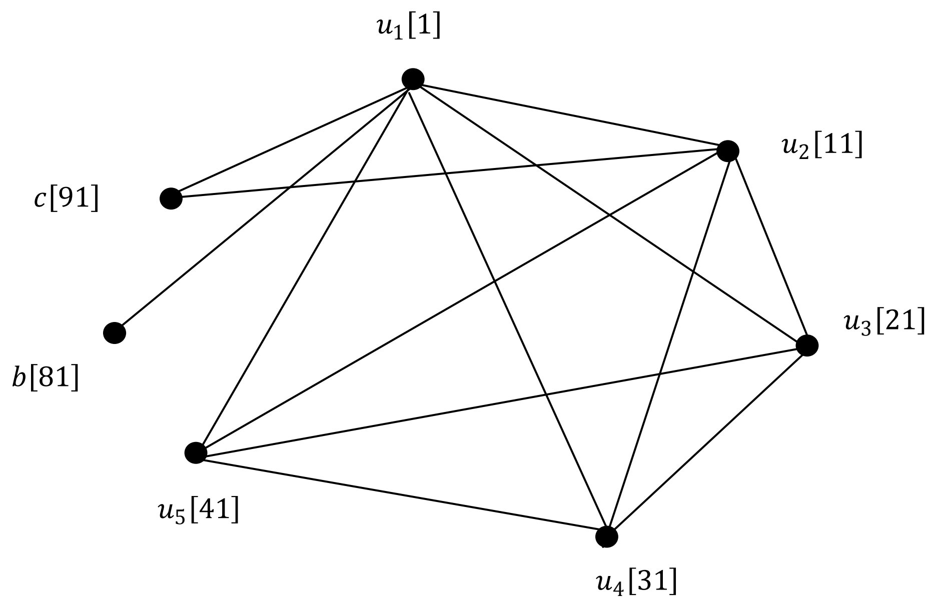
Let $1\le r\le n-1,$

$c+u_j=(n+r+j-1)\times 10+2\in S\leftrightarrow 1\le n+r+j-1\le 2n\Leftrightarrow 1\le j\le n-r+1$

and $b+u_j=(n+r+j-2)\times 10+2\in S\leftrightarrow 1\le n+r+j-2\le 2n\Leftrightarrow 1\le j\le n-r+2$ .Then $V\left(2K_{1,r}\right)\setminus\{b\}\cup\{c\}=\{u_{n-r+1},u_{n-r+2},...u_n\}$. So it is sum labeling of $\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)\cup(2n)K_1$ .Thus $\sigma\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)\le 2n$.□

We have the following Theorem by Lemma 2.9-2.11.

Theorem2. For $n\ge 4, 2\le r\le n-1, \sigma\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)=\zeta\left(K_{n+2}\setminus E\left(2K_{1,r}\right)\right)=2n$.



- 12
- 22
- 32
- 42
- 52
- 62
- 72
- 82
- 92
- 102

Fig-1 sum and integral sum number of a graph $\left(K_7 \setminus 2K_{1,4}\right)$=10

REFERENCES

[1] Chen, Z. (1996), F. Harary'sConjecture On Integral Sum Graphs, Discrete Math., Vol. 160, pp. 241-244.

[2] Harary, F. (1994), Sum Graph Over All Integers, Discrete Math., Vol. 124, pp. 99-105.

[3] Harary, F. (1990), Sum Graph and Difference Graphs,Cong.Numer., Vol. 72, pp. 101-103.

[4] Liaw, S. C., Kuo, D. and Chang, G. J. (2000), Integral Sum Numbers of Graphs, ArsCombin., Vol. 54, pp. 259 - 268.

[5] Sharary, A. (1996), Integral Sum Graphs From Complete Graphs, Cycles And Wheels, Arab Gulf j.Sci.Res., Vol. 14, Issue 1, pp. 1-14.

[6] Wang, H. (2009), The Sum Numbers And Integral Sum Numbers Of Graph $K_{n+1} \setminus E\left(K_{1.r}\right)$, Discrete Math., Vol. 309, pp. 4137- 4143.

[7] Wang, Y. and Liu, B. (2001), The Sum Number And Integral Sum Number Of Complete Bipartite Graph, Discrete Math., Vol. 239, pp. 69 - 82.

[8] He, W., Yu, X., Mi, H., Xu, Y., Sheng, Y. and Wang, L. (2002), The(Integral) Sum Number Of $K_n - E\left(K_r\right)$, Discrete Math., Vol. 243, pp. 241-252.