On Symmetric Solutions of Elliptic Boundary Value Problems

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Abstract: **The aim of this paper is to study the symmetry of solutions of nonlinear elliptic differential equations of type $\Delta u + V(|x|)e^u = 0$ in R^3, which arise in geometry and various branches of physics. Symmetry of solutions is proved by applying method of moving planes.**

Keywords- Maximum principles; Moving plane method; Elliptic boundary value problems.

I. INTRODUCTION

It is an important goal in mathematical analysis to establish symmetry properties of symmetry properties of solutions of boundary value problems both from theoretical as well as the application point of view. To prove the symmetry J. Serrin [17] introduced the method of moving planes in the differential equations. It has been previously used by A. D. Alexandroff [12] in differential geometry. In 1979 the same method was introduced by Gidas, Ni and Nirenberg [8],[9] to obtain the symmetry results and monotonicity for positive solutions of nonlinear elliptic equations. Yi Li and Wei-Ming Ni [13] proved the symmetry results for the conformal scalar curvature equation and Matukuma equation. Guo and Wei [11] use the moving plane method to obtain the necessary and sufficient conditions for the radial symmetry of positive solution of semilinear equation with singular nonlinearity. Recently Dhaigude and Patil [5], [6] [7] studied radial symmetry of positive solutions of semilinear elliptic boundary value problem in unit ball and in R^n, by using moving plane method. In this paper we study the radial symmetry of classical solutions for semilinear elliptic boundary value problems of the type,

$$\Delta u + V(|x|)\, e^u = 0 \quad \text{in } R^3 \qquad 1.1$$

$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty \qquad 1.2$$

These types of problems are studied by Yuki Naito [15] in R^2. Problems of this kind arise in geometry and various branches of physics , see Chanillo and Kiessling [1]. In the case where V is a constant coefficient, we refer to Chen and Li [3]. In the case where $V(r)$ is a variable coefficient, we refer to Chen and Li [4]. Also Chen and Li [2] explained and used moving plane method in proving symmetry. We organise the paper as follows: In section 2, the method of moving plane is explained. In section 3, the preliminary results and some useful lemmas are proved. The symmetry result and corollaries are proved in the last section.

II. MOVING PLANE METHOD

We use the moving plane method as follows.

Suppose that R^n is an Euclidean space. Let $u(x)$ be a positive solution of a certain partial differential equation in R^n. To prove $u(x)$ is symmetric and monotone in the given direction, assign that direction as X_1 axis. We define $T_\lambda = \{x = (x_1, x_2, ..., x_n): x_1 = \lambda\}$for a real number λ. This is the plane perpendicular to X_1 -axis and it is the plane that we will move with X_1 -axis.

Let Σ_λ denote the region to the left of the plane in Ω i.e. $\Sigma_\lambda = \{x : x_1 < \lambda , x \in \Omega \}$. The reflection of the point $x = (x_1, x_2, ..., x_n)$, about the plane $T\lambda$ is denoted by x^λ and it is

$x^\lambda = (2\lambda - x_1 , x_2, x_3, ..., x_n)$. We compare the values of $u(x)$ at x and x^λ. To show that $u(x)$ is symmetric about plane T_λ, We have to show that $u(x) = u(x^\lambda)$. For this suppose that

$$w_\lambda(x) = u(x) - u(x^\lambda).$$

To show that there exist some λ_0 such that $w_{\lambda 0}(x) = 0$ for all $x \in \Sigma_{\lambda 0}$. We consider following steps.

Step-I: We first show that for λ sufficiently negative we have, $w_\lambda(x) \geq 0$ for all $x \in \Sigma_\lambda$.

Then we are able to start of from the neighbourhood of $x_1 = -\infty$ and move the plane T_λ along the x_1 direction to the right as long as $w_\lambda(x) \geq 0$ holds for all $x \in \Sigma_\lambda$.

Step-II: We continuously move this plane up to it's limiting position.

Define $\lambda_0 = \sup \{\lambda : w_\lambda(x) \geq 0 \text{ for all } x \in \Sigma_\lambda \}$. We prove that u is symmetric about the plane $T_{\lambda 0}$. i.e. $w_{\lambda 0}(x) = 0$ for all $x \in \Sigma_{\lambda 0}$. This is usually carried out by the method of contradiction. We suppose $w_{\lambda 0}(x) \neq 0$, then there exist $\lambda > \lambda_0$ such that $w_{\lambda 0}(x) < 0$. This is contradiction to the definition of λ_0. From this we can see that key to the method of moving plane is to establish the inequality $w_\lambda(x) \geq 0$ for all $x \in \Sigma_\lambda$.

Before proceeding to the main result we shall set forth some preliminaries and hypotheses.

2.1 Preliminary Results.

Lemma 2.1 [14] Let Ω be unbounded domain in R^3. Suppose that $u \neq 0$ satisfies $L(u) \leq 0$ in Ω and $u \geq 0$ on $\partial\Omega$

where

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}(u) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}(u) + c(x)u.$$

Suppose furthermore that there exist a function w such that $w > 0$ on $\Omega \cup \partial\Omega$ and $L(w) \leq 0$ in Ω.

If $\frac{u(x)}{w(x)} \to 0$ as $|x| \to \infty$, $x \in \Omega$, then $u > 0$ in Ω.

Theorem 2.1

Let u(x) satisfies differential inequality $L(u) \geq 0$ in a domain D where L is uniformly elliptic. If there exist a function w(x) such that $w(x) > 0$ on $D \cup \partial D$,

$L(w) \leq 0$ in D,

then $\frac{u(x)}{w(x)}$ cannot attain a non negative maximum at a point p on ∂D, which lies on the boundary of a ball in D and $\frac{u(x)}{w(x)}$ if is not constant then, $\frac{\partial}{\partial \nu}\left(\frac{u}{w}\right) > 0$ at P, where $\frac{\partial}{\partial \nu}$ is any outward directional derivative.

Lemma 2.2 [16] {Hopf Boundary lemma} : Suppose that Ω satisfies the interior sphere condition at $x_0 \in \Omega$. Let L be strictly elliptic with $c \leq 0$. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $L(u) \geq 0$ and $\max_{\overline{\Omega}} u(x) = u(x_0)$ then either $u = u(x_0)$ on Ω 2.1

or

$$\liminf_{t \to 0} \left(\frac{u(x) - u(x_0 + t\nu)}{t} \right) > 0$$

for every direction ν, pointing into an interior sphere.

If $u \in C^1 \subset \Omega \cap \{0\}$ then,

$$\frac{\partial u}{\partial \nu}(x_0) < 0$$

III. MAIN RESULT

In this section, first we prove some lemmas which are useful to prove our main result. We define the following function,

$$w(x) = \frac{1}{3\omega_3} \int_{R^3} \left(\frac{1}{|x-y|} - |y| \right) f(y)dy \qquad 3.1$$

Lemma 3.1 Let $f \in L^\infty(R^3) \cap L^1(R^3)$ Let w be as defined in 3.1. Then

$$\lim_{|x| \to \infty} \frac{w(x)}{\log|x|} = \frac{1}{3\omega_3} \int_{R^3} f(y)dy \qquad 3.2$$

Proof: To prove the result,

$$\lim_{|x| \to \infty} \frac{w(x)}{\log|x|} = \frac{1}{3\omega_3} \int_{R^3} f(y)dy$$

we have to prove that,

$$\lim_{|x| \to \infty} \int_{R^3} \frac{\frac{1}{3\omega_3}\left(\frac{1}{|x-y|} - |y| \right)}{\log|x|} f(y)dy = \frac{1}{3\omega_3} \int_{R^3} f(y)dy$$

To complete the proof it is sufficient to prove that,

$$\int_{R^3} \left(\frac{1}{\log|x|\,|x-y|} - \frac{|y|}{\log|x|} - 1 \right) f(y)dy \to 0 \text{ as } |x| \to \infty.$$

We know that $f \in L^\infty(R^3)$ means for any $\epsilon > 0$ there exist $R > 0$ such that

$$\int_{|y|>R} |f(y)|dy < \epsilon \qquad 3.3$$

For simplicity we divide the region of integration in to three parts D_1, D_2 and D_3 where

$$D_1 = \{ y \in R^3 : |y - x| \leq 1 \}$$

$$D_2 = \{ y \in R^3 : |y - x| > 1 \text{ and } |y| \leq R \}$$

$$D_3 = \{ y \in R^3 : |y - x| > 1 \text{ and } |y| > R \}$$

All the sub regions D_1, D_2 and D_3 forms partition on R^3. Therefore,

$$I = I_1 + I_2 + I_3 \qquad 3.4$$

where I_1, I_2 and I_3 are integrals over the region D_1, D_2 and D_3 respectively.

We have to show that $I \to 0$ as $|x| \to \infty$.

For this purpose we shall determine I_1, I_2 and I_3 separately

$$I_1 = \left| \int_{D_1} \left(\frac{1}{\log x\ |x-y|} - \frac{|y|}{\log|x|} - 1 \right) f(y)dy \right|$$

$$\leq \int_{D_1} \left| \left(\frac{1}{\log x\ |x-y|} - \frac{|y|}{\log|x|} - 1 \right) \right| |f(y)|dy$$

$$\leq \int_{D_1} \left| \frac{1}{\log x\ |x-y|} |f(y)| \right| dy + \int_{D_1} \left(\frac{|y|}{\log|x|} + 1 \right) \left| \ |f(y)|dy \right.$$

$$\leq \int_{D_1} \frac{1}{\log x\ |x-y|} |f(y)|dy + \int_{D_1} C\ |f(y)|dy$$

$$\leq \int_{D_1} \frac{1}{\log x\ |x-y|} |f(y)|dy + C \int_{D_1} |f(y)|dy$$

$$\leq \frac{||f(y)||_{L^\infty}}{\log|x|} \int_{D_1} \frac{1}{|x-y|}\, dy\, C \int_{D_1} |f(y)|dy$$

where

C = $\frac{|y|}{\log|x|} + 1$

We evaluate

$$\int_{D_1} \frac{1}{|x-y|}\, dy$$

where D_1 is a sphere $|x - y| \leq 1$.

we have

$$\int_{D_1} \frac{1}{|x-y|}\, dy = 2\pi$$

Therefore $|I_1| \leq C\epsilon$

To estimate I_2, we obtain

$$|I_2| = |\int_{D_2} \left(\frac{1}{\log x\ |x-y|} - \frac{|y|}{\log|x|} - 1\right) f(y)dy\, |$$

$$\leq \int_{D_2} \left|\left(\frac{1}{\log x\ |x-y|} - \frac{|y|}{\log|x|} - 1\right)\right| |f(y)|dy$$

$$\frac{||f(y)||_{L^\infty(R^3)}}{\log|x|} \left[\int_{D_2} \frac{1}{|x-y|} - \log|x| - |y|\right] dy$$

We note that $|\frac{1}{|x-y|} - \log|x|| < C$, for $|y| \leq R$

and $\int_{|y|\leq R} |y|dy < \infty$

So I_2 tends to zero as $|x| \to \infty$.

To evaluate I_3 , consider

$$|I_3| = |\int_{D_3} \left(\frac{1}{\log x\ |x-y|} - \frac{|y|}{\log|x|} - 1\right) f(y)dy\, |$$

$$\leq \int_3 \left|\left(\frac{1}{\log x\ |x-y|} - \frac{|y|}{\log|x|} - 1\right)\right| |f(y)|dy$$

Since in D_3 $|x - y| > 1$ and $|y| > R$

$|I_3| < C\epsilon$.

Thus $|I| \leq 2C\epsilon$.

Since I_1, I_3 are finite integrals and $I_2 = 0$ their sum tends to zero as $|x|\to\infty$.Thus $I \to 0$ as $|x|\to\infty$.

Hence the result is obtained.

Lemma 3.2 (Liouville's theorem) Assume that w be harmonic function in R^3 and satisfies $w(x) = o(|x|)$ as $|x| \to \infty$. Then w must be constant.

Proof: To prove w is constant, prove that Dw = 0**.** Fix x_0 in R^3. Define

$$B_r(x_0) = \{y \in R^3 : |y - x_0| < r\}$$

For some r > 0. Since the gradient Dw is also harmonic function in R^3, it follows by mean value and divergence theorems that

$$Dw(x_0) = \frac{1}{4\pi r^3} \int_{B_r(x_0)} Dw.dx$$

$$= \frac{1}{\omega_3(R^3)} \int_{\partial B_r(x_0)} w.\nu\, ds$$

where ν is outward unit normal to the surface $\partial B_r(x_0)$.

$$|Dw(x_0)| = |\frac{1}{\omega_3(R^3)} \int_{\partial B_r(x_0)} w.\nu\, ds|$$

$$\leq \frac{3}{R^2} \underbrace{Sup}_{\partial B_r(x_0)} |w| \to 0 \text{ as } R \to \infty$$

$$\therefore |Dw| \to 0 \text{ as } R \to \infty$$

$$\therefore |Dw| = 0 \text{ in } R^3$$

So w is constant.

Define

$\Lambda = \{\lambda \in (0,\infty): V_\lambda(x) > 0\}$ for $x \in \Sigma_\lambda$.

Lemma 3.3 Let u be solution of 1.1 satisfying $u^+ \in L^\infty(R^3)$ and

$$0 \leq \frac{1}{3\omega_3} \int_{R^3} V(|x|)e^u\, dx = \beta < \infty$$

then $\lim_{|x|\to\infty} \frac{u(x)}{\log|x|} = \lim_{|x|\to\infty} \frac{u(x^\lambda)}{\log|x|} = \beta$

Proof: Define the function

$$w(x) = \frac{1}{3\omega_3} \int_{R^3} \left(\frac{1}{|x-y|} - |y|\right) V(y)e^{u(y)}\, dy \text{ in } R^3 \quad (3.6)$$

The function w(x) is well defined and by lemma in [11] we have

$\Delta w = V(|x|)e^u$ in R^3.

From equation (3.5) and $u^+ \in L^\infty(R^3)$

$ve^u \in L^\infty(R^3) \cap L^1(R^3)$

Then by lemma 3.1

$$\lim_{|x|\to\infty} \frac{w(x)}{\log|x|} = \frac{1}{3\omega_3} \int_{R^3} v(y)e^u dy = \beta$$

Consider the function

u(x) + w(x) = v(x)

Operating Δ on both sides, we have

$$\Delta u(x) + \Delta w(x) = \Delta v(x) \quad (3.7)$$

We have $\Delta u(x) = -\Delta v(x)e^{u}$ and

$$\Delta w(x) = v(|x|)e^{u}$$

$\Delta v(x) = 0$ in R^3.

Also v(x) is o(|x|) , as |x|→∞ . By lemma 3.2 we know that v is constant say c. Divide equation (3.7) by log |x| and taking limit of both side as |x| →∞ we get

$$\lim_{|x|\to\infty} \frac{c}{\log|x|} = \lim_{|x|\to\infty} \frac{u(x)}{\log|x|} + \lim_{|x|\to\infty} \frac{w(x)}{\log|x|}$$

$$0 = \lim_{|x|\to\infty} \frac{u(x)}{\log|x|} + \lim_{|x|\to\infty} \frac{w(x)}{\log|x|}$$

$$\lim_{|x|\to\infty} \frac{u(x)}{\log|x|} = -\lim_{|x|\to\infty} \frac{w(x)}{\log|x|} = -\beta$$

Also

$$\lim_{|x|\to\infty} \frac{u(x^{\lambda})}{\log|x|} = -\beta$$

Suppose that V(r) is locally Hölder continuous function on [0, ∞) and that V(r) is nonincreasing in r ≥ 0.

Lemma 3.4 Let **λ > 0** then V_λ satisfies,

$$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \le 0 \quad \text{in } \Sigma_\lambda$$

where $C_\lambda(x)$ satisfies $C_\lambda(x) = O(|x|^{-\delta})$ as |x| →∞ for some $\delta \ge 3$.

Proof: Let V(r) is nonincreasing in r and $|x^\lambda| > |x|$ for $x \in \sum_\lambda$ and λ > 0 we have

$$0 = \left(\Delta u(x) + V(|x|)e^{u(x)}\right) - \left(\Delta u(x^\lambda) + V(|x|^\lambda)e^{u(x^\lambda)}\right)$$

$$= \Delta(u(x) - u(x^\lambda)) + V(|x|)e^{u(x)} - V(|x|^\lambda)e^{u(x^\lambda)}$$

$$\ge \Delta(V_\lambda)(x) + V(|x|)\left(e^{u(x)} - e^{u(x^\lambda)}\right) \ge \Delta(V_\lambda)(x) + C_\lambda(x)V_\lambda(x)$$

where,

$$C_\lambda(x) = V(|x|)e^{u(x^\lambda)+t(u(x)-u(x^\lambda))}dt$$

Take $\epsilon > 0$ so small that $\alpha + \beta - \epsilon > 3$. By lemma 3.3 we have, $\frac{u(x)}{\log|x|} \le -(\beta - \epsilon)$.

Therefore, $u(x) \le -\log|x|\,(\beta - \epsilon)$.

Also, $u(x^\lambda) \le -\log|x|(\beta + \epsilon)$

We have $\limsup_{r\to\infty} r^\alpha V(r) < \infty$ for $\alpha > 0$.

$$\therefore C_\lambda(x) = V(|x|)\int_0^1 e^{u(x^\lambda)+t(u(x)-u(x^\lambda))}dt$$

$\therefore C_\lambda(x) = O(|x|)^{-\delta}$ where $\alpha + \beta - \epsilon = \delta$ as |x|→∞.

Remark 3.1 By virtue of lemma 3.4 we can take $R_0 > \epsilon$ so large that $\frac{1}{1+|x|\log\frac{1}{|x|}} \ge C_\lambda(x)$

Lemma 3.5 Let $\lambda > 0$. If $V_\lambda > 0$ on $\sum_\lambda \cap \overline{B_{R_0}}$, then $\lambda \in \Lambda$.

Proof: By lemma 3.4 and assumption we have
$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \le 0$ in $\Sigma_\lambda \setminus \overline{B_{R_0}}$

$V_\lambda \ge 0$ on $\partial(\Sigma\lambda \setminus \overline{B_{R_0}})$

Let $w(x) = \frac{1}{|x|} - \log|x|$. Then w(x) satisfies

$$\Delta w + \frac{1}{|x|} = 0$$

$\therefore \Delta w + C_\lambda(x)w \le 0$ in $\Sigma_\lambda \setminus \overline{B_{R_0}}$

By lemma 3.3, $w > 0$ in $\overline{\Sigma_\lambda \setminus B_{R_0}}$

$\therefore \frac{v_\lambda(x)}{w(x)} \to 0$ as |x| →∞

By lemma 2.1 we have

$V_\lambda(x) > 0$ in $\Sigma_\lambda \setminus \overline{B_{R_0}}$.

$V_\lambda(x) > 0$ in Σ_λ.

$$\therefore \lambda \in \Lambda.$$

Lemma 3.6 Let $\lambda \in \Lambda$ then $\frac{\partial u}{\partial x_1} < 0$
Proof: By lemma 3.4 we have

$\Delta V_\lambda(x) + C_\lambda(x)V_\lambda(x) \le 0$ in Σ_λ.

Since $V_\lambda = 0$ on T_λ,

$\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ. (By Hopf boundary maximum lemma)

$\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial V_\lambda}{\partial x_1} < 0$ on T_λ.

Now we will state and prove main theorem about symmetry.

Theorem 3.1 Assume that V satisfies,

$$\limsup_{r\to\infty} r^\alpha . V(r) < \infty \text{ for some, } \alpha > 0 \quad (3.9)$$

Let u be the solution of [1.1] satisfying $u \in L^\infty(R^3)$, where $u^+ = \max\{u, 0\}$ and

$$0 < \frac{1}{3\omega_3}\int_{R^3} V(|x|)e^u dx = \beta < \infty \quad (3.10)$$

If $\alpha+\beta>3$ then u must be radially symmetric and decreasing. Furthermore, assume that V is not constant then u must be radially symmetric about origin and $u_r<0$ for r > 0.

Proof: Let u be the solution of [1.1] satisfying $u^+ \in L^\infty(R^3)$. and equation (3.10) . From lemma 3.3 we have,

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|} = -\beta$$

$$\therefore \lim_{|x|\to\infty} u(x) = -\infty$$

Then there exist $R_1 > R_0$ such that

$$\max\{u(x): |x| > R_1\} \leq \min\{u(x): |x| \leq R_0\}$$

where R_0 is constant such that $|x| > R_0$ we have $\frac{1}{1+|x|\log\frac{1}{|x|}} \geq C_\lambda(x)$. We shall prove the theorem in following three steps.

Step-I: To prove $[R_1, \infty) \subset \Lambda$. Let $\lambda \in [R_1, \infty)$, so $\lambda \geq R_1$. We note that $\overline{B_0} \subset \sum_\lambda$. But $V_\lambda > 0$ in B_0. Then by lemma 3.5 $\lambda \in \Lambda$. It implies that $[R_1, \infty) \subset \Lambda$.

Step-II: Let $\lambda_0 \in \Lambda$, then there exist $\epsilon > 0$ such that $(\lambda_0-\epsilon, \lambda_0] \subset \Lambda$. Assume to the contrary that there exist an increasing sequence $\{\lambda_i\}$, such that $\lambda_i \to \lambda_0$ as $i\to\infty$. By lemma 3.5 we have a sequence $\{x_i\}$, i = 1,2,3,... such that $x_i \in \sum_{\lambda_i} \cap \overline{B_{R_0}}$ and $V_{\lambda_i}(x_i) \leq 0$. A subsequence which we call again sequence $\{x_i\}$ converges to some point $x_0 \in \sum_{\lambda_i} \cap B_{R_0}$. Then $V_{\lambda_0}(x_0) \leq 0$. Since $V_{\lambda_0} > 0$ in $\sum_{\lambda_0}$, we have $x_0 \in T_{\lambda_0}$. By mean value theorem there exist a point y_i satisfying $\frac{\partial u}{\partial x_1}(y_i) \geq o$ on the straight line segment joining x_i to $(x_i)^{\lambda_i}$ for each i = 1,2,3,... . Since $y_i \to x_0$ as $\to\infty$, we have $\frac{\partial u}{\partial x_1}(x_0) \geq 0$. On the other hand, since $x_0 \in T_{\lambda_0}$, we have $\frac{\partial u}{\partial x_1}(x_0) < 0$. This is a contradiction, hence step II is proved.

Step-III: We have to prove either statement (A) or statement (B) holds.

(A) $u(x) = u(x^\lambda)$ for some $\lambda_1 > 0$ and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for some $\lambda > \lambda_1$.

(B) $u(x) = u(x^0)$ in σ_0 and $\frac{\partial u}{\partial x_1} < 0$ on T_λ for some $\lambda > 0$.

Let $\lambda_1 = \inf\{\lambda > 0 : [\lambda, \infty) \subset \Lambda\}$. We distinguish the proof in two cases: (i) $\lambda_1 > 0$ and (ii) $\lambda_1 = 0$.

Case (i): Let $\lambda_1 > 0$.

Let $V_{\lambda 1}(x) = u(x) - u(x^{\lambda 1})$. Since u is continuous we have $V_{\lambda 1}(x) \geq 0$ in $\Sigma_{\lambda 1}$. From lemma 3.4 we have

$$\Delta V_{\lambda_1}(x) + C_{\lambda_1}(x)V_{\lambda_1}(x) \leq 0 \text{ in } \Sigma_{\lambda_1}.$$

Hence by strong maximum principle we have that either $V_{\lambda_1} > 0$ in Σ_{λ_1} or $V_{\lambda_1} = 0$ in Σ_{λ_1} .

Assume that $V_{\lambda_1} > 0$ in Σ_{λ_1} then $\lambda_1 \in \Lambda$. From *step* (II) there exist $\epsilon > 0$ such that

$(\lambda_1-\epsilon, \lambda] \subset \Lambda$. This contradicts to the |definition of λ_1. So $V_{\lambda_1}(x) = 0$. Since $(\lambda_1, \infty) \subset \Lambda$ we have $\frac{\partial u}{\partial x_1} < 0$ on T_λ for $\lambda > \lambda_1$. (By lemma 3.6) .Thus we get statement (A).

Case(ii): Let $\lambda_1 = 0$.

From continuity of u. $V_{\lambda 0}(x) \geq 0$ in Σ_0. By lemma 3.6 we have $\frac{\partial u}{\partial x_1} < 0$ *on* T_λ for $\lambda > 0$. Assume that V(r) is not constant, in this case we have to prove that (A) holds. From (1.1) we have $V(|x|) = V(|x^\lambda|)$ for $x \in \Sigma_\lambda$.

Since V(r) is nonincreasing, we have w is constant. This contradicts to the assumption. Thus (B) holds. If (B) occurs in step (III) then we can repeat all the three steps for negative X_1 –direction about plane $x_1 = \lambda_1 < 0$ or

$$u(x) \leq u(x_0) \text{ in } \Sigma_0. \qquad (3.11)$$

If (3.11) occurs then $u(x) = u(x^0)$ in Σ_0. Therefore u must be radially symmetric in X_1 –direcion about some plane and strictly decreasing away from the plane. Since equation (1.1) is invariant under rotation we may take any direction as X_1 –direction and conclude that u is symmetric in every direction about some plane. Therefore, u is radially symmetric about origin and $u_r < 0$ for r > 0.

We give some corollaries of the theorem. First we consider the case where $V(r)$ is nonpositive for r large. In this case we take $\alpha > 3$ in (3.9) and can obtain the following.

Corollary-1: Suppose that V(r) is nonpositive for large R. Let u be the solution of (1.1) satisfying $u^+ \in L^\infty(R^3)$. Then u must be radially symmetric about the origin and $u_r < 0$ for r > 0.

Next we consider the case where the V(r) is nonnegative for $r \geq 0$.

Corollary-2: Suppose V(r) is nonnegative for $r \geq 0$ and satisfies $\limsup_{r\to\infty} r^\alpha V(r) < \infty$, with $\alpha > 3$. Let u be the solution of (1.1) satisfying $u^+ \in L^\infty(R^3)$. Then u must be radially symmetric about the origin and $u_r < 0$ for r > 0.

Corollary-3: Suppose V(r) is nonnegative for $r \geq 0$ Let u be the solution of (1.1) satisfying $\int_{R^3} e^u\, dx < \infty$ (3.12)

Then u must be radially symmetric and decreasing.

Corollary-4: Suppose V(r) is nonnegative for $r \geq 0$ Let u be the solution of (1.1) satisfying $\frac{u(x)}{\log|x|} \to -\beta$ as $|x| \to \infty$ with $\beta > 3$. (3.13)

Then u must be radially symmetric and decreasing.

Here we see that u satisfies (3.12). As a consequence of corollary [3] we obtain corollary [4].

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