A Study of Fluid Filtration in Capillary Tissue Fluid Exchange System

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Abstract:- **The aim of our present work is to develop a model for capillary tissue fluid exchange system and to investigate the effect of permeability of the tissue and the parameters related to the casson's fluid model for the blood on the fluid filtration and capillary and tissue pressure. The model consists of single capillary surrounded by porous tissue. The blood in capillary is modeled by casson's fluid with peripheral layer. The governing equations in the tissue as well as in capillary region are solved using analytical method and computational results have been obtained. It has been observed that filtration decreases as the viscosity of the peripheral layer increases.**

I. INTRODUCTION

All the fluid outside the cells are collectively called the extracellular fluid. The two largest compartments of the extracellular fluid are interstitial fluid which makes up about three fourths of the extracellular fluid and the plasma, which makes up almost one fourth of the extracellular fluid. The plasma is the noncellular part of the blood and communicated continuously with the interstitial fluid through the pores of the capillary membrane. These pores are highly permeable to almost all solutes in the extracellular fluid except the proteins. Therefore the extracellular fluids are constantly mixing, so that the plasma and the interstitial fluids have about the same composition except for proteins which have a higher concentration in the plasma.

Edema refers to the presence of excess fluid in the body tissue. In most instances edema occurs mainly in the extracellular fluid compartment, but can involve intracellular fluids as well. Extracellular fluid edema occurs when there is excess fluid accumulation in the extracellular spaces. There are two general causes of extracellular edema (i) abnormal leakage of fluid from the plasma to the interstitial spaces across the capillaries and (ii) failure of the lymphatic to return fluid from the interstitium back into the blood. The most common clinical cause of interstitial fluid accumulation is excessive capillary fluid filtration. A large number of conditions can cause fluid accumulation in the interstitial space by abnormal leakage of fluid from the capillaries or by preventing from returning fluid from the interstitium back to the circulation.

There are three major factors that cause increased capillary filtration of fluid and protein into the interstitium. i) increased capillary hydrostatic pressure (ii) decreased plasma colloid osmotic pressure and (iii) increased capillary permeability, which causes leakage of proteins and fluid through the pores of the capillaries.

Therefore the studies of movement of body fluids across membrane and through tissue compartment are basic physiological problems. There are lot of challenging problems to be investigated. Mathematical analysis of these movements would be highly complex without the use of simplifying models and assumptions about the microcirculation.

Starting from the first model of Krogh known as Krogh cylinder model of capillary tissue oxygen transport model. Through a series of research papers by research group led by Fitzgerald, Secomb and Skalak have separately developed the subject for closely filtered cells in capillaries through which cells enter in various deformed shapes and plasma squeezes through the small gaps between the cell and the capillary. Nappier and Shubert, Salathe et.al. and Secomb et.al. have separately developed models of oxygen transport through single capillary and whole tissue organs. Salathe and Venkatraman have discussed model of capillary tissue fluid exchange and role of extravascular protein on fluid exchange. They have also considered the interaction of fluid movement and particle diffusion across capillary walls. The rate of transcapillary exchange of substances delivered to or removal from the tissue by blood depends on concentration difference across the capillary wall. Salathe developed a model of capillary tissue fluid exchange. This model includes the mutual interaction of fluid movement across the capillary wall and convection and the diffusion of a number of solutes also developed a mathematical model of blood flow in a coronary capillary. In view of the non-Newtonian nature of blood in capillaries and filtration /absorption property of the wall, Oka (1979) studied blood flow in capillaries with permeable walls using the casson fluid model.

A survey of the literature on the topic indicates that barring a few who discussed the problem for filtration efficiency in the capillary tissue exchange system. As

discussed earlier that moment of fluids across tissue may affect the fluid properties and influence the flow patterns, particularly small diameter vessels. A study of filtration efficiency in the capillary tissue exchange system may be of significant practical use and clinical application. It appears that no effort at least to the author's knowledge has been made to observe the filtration efficiency representing blood as three layer model. Motivated by above studies we have developed a mathematical model for capillary tissue fluid exchange system. A Krogh cylinder model has been taken to represents the capillary surrounded by tissue with peripheral layer. Blood is treated as two fluid model with the suspension of all the erythrocytes in the core region as a casson fluid and the plasma in the peripheral region as a Newtonian fluid Tissue is represented by porous matrix which is governed by Darcy's Law. Under admissible assumptions the governing equation in two regions has been solved by using proper boundary and interfacial conditions with the help of mathematical techniques. Filtration from a capillary into the surrounding tissue and flow in the capillary has been analyzed

II. FORMULATION OF PROBLEM

Figure1 shows the flow geometry corresponding to the cylindrical polar coordinate system, where r and x denote the radial and axial coordinates respectively. The problem has been investigated under the following assumptions:

Assumptions:

Whole blood is a complex mixture and a capillary is comparable in diameter to red cells. Therefore an attempt to analyse the system in formal manner is very difficult and then we have to use number of simplifications: **i)** The capillary tube between an arteriole and venule is a straight tube of a uniform circular cross section as in Fig.1.**ii)** The blood is non Newtonian fluid. **iii)** The motion of blood is laminar, slow axisymmetric and steady **iv)** No body force acts on the blood.

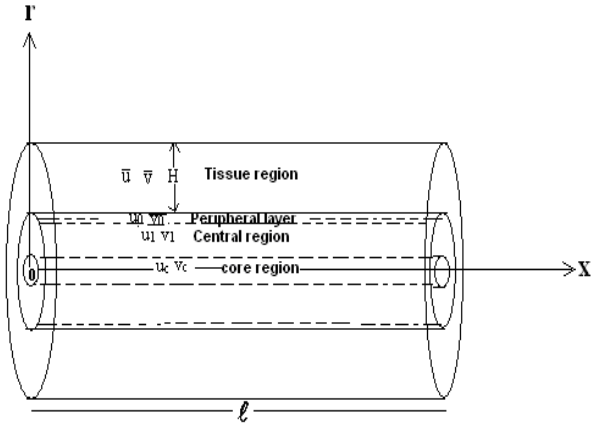

Fig.1 Schematic diagram

The whole region is divided into two regions

1. Capillary Region
2. Tissue Region

1. Capillary Region: We have considered the axisymmetric flow of blood in a cylindrical capillary surrounded by tissue. In capillary it is assumed that blood is represented by a two layered model with a central layer of casson fluid of radius R_1 and a peripheral layer of plasma of radius R_2. Core region is assumed to be uniform hematocrit of radius r_h.

Under the admissible assumptions the equation of motion and the continuity for peripheral layer, central region and core region is written as below:

(i) For Peripheral layer region

$$-\frac{\partial P'}{\partial x'}+\frac{\mu_0}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial u_0'}{\partial r'}\right)=0 \qquad (1)$$

$$(R_1 < r < R_2)$$

$$-\frac{\partial u_0'}{\partial x'}+\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'v_0'\right)=0 \qquad (2)$$

(ii) For central region

$$\tau_1^{1/2}=\tau_0^{1/2}+\mu_1^{1/2}\left(-\frac{\partial u_1}{\partial r}\right)^{1/2} \qquad \tau_1 \geq \tau_0 \qquad (3)$$

$$\frac{\partial u_1}{\partial r}=0 \qquad \tau_1 \leq \tau_0 \qquad (4)$$

Where τ_1 is stress in central region. τ_0 is constant yield stress in the core region. $\frac{\partial u_1}{\partial r}$ is the rate of strain of the casson fluid. μ_1 denotes casson's viscosity. These relations corresponds to vanishing of velocity gradients in the region where the shear stress τ_1 is less than yield stress τ_0. This implies plug flow, whenever $\tau_1 \leq \tau_0$

Considering the forces on the control volume and equating the shear forces and pressure forces acting on the control volume, we get

$$2\pi(r+dr)\ell(\tau_1+\delta\tau_1)-2\pi r\ell\tau=-\frac{\partial P}{\partial x}2\pi r\ell\delta r \qquad (5)$$

Dividing by δr and taking the limit as $\delta r \to 0$, we get

$$\frac{d}{dr}(r\tau)=-r\frac{\partial P}{\partial x}$$

which on integration leads to

$$\tau_1 = -\frac{\partial P}{\partial x}\frac{r}{2} + \frac{A}{r}$$

The constant A is determined by the condition that τ_1 is finite at $r = 0$
and we get

$$\tau_1 = \frac{r}{2}P$$

Where $P = -\frac{\partial P}{\partial x}$ is the pressure gradient of the flow in capillary.

again putting $r = r_h$, $\tau_1 = \tau_0$ we get

$$\tau_0 = \frac{r_h}{2}P$$

$$-\frac{\partial P'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\tau_1'\right) = 0 \qquad (6)$$

$$\left(r_h < r < R_1\right)$$

$$-\frac{\partial u_1'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'v_1'\right) = 0 \qquad (7)$$

Where $\tau_1^{1/2} = \tau_0^{1/2} + \mu_1^{1/2}\left(-\frac{\partial u_1}{\partial r}\right)^{1/2}$

2. Tissue Region: Governing equation in tissue region is given by Darcy's law

$$\bar{u}' = -\frac{K}{\mu}\frac{\partial \bar{P}'}{\partial x'},\ \bar{v}' = -\frac{K}{\mu}\frac{\partial \bar{P}'}{\partial r'} \qquad (8)$$

$$\frac{\partial \bar{u}'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\bar{v}'\right) = 0 \qquad (9)$$

where x' and r' are the axial and radial coordinates, P' and $\bar{P}'$ are the pressure in capillary region and tissue region respectively. u_0' v_0', u_c' v_c', u_1' v_1' and $\bar{u}'$ $\bar{v}'$ are the axial and radial component of velocity in cell free layer, core region, central region and tissue region. K is the permeability of tissue. μ_0 μ_1 μ_c and μ are the viscosity of cell free layer, central region, core region and tissue region.

III. BOUNDARY AND MATCHING CONDITIONS

τ is finite at $r' = 0$

Slip velocity is assumed at the porous boundary

$$u_0' - \bar{u}' = -\sigma'\frac{\partial u_0'}{\partial r'} \text{ at } r' = R_2$$

The velocity is continuous at the interface of plasma and the core

$$u_0' = u_1' \text{ at } r' = R_1'$$

No flux condition is assumed at the outer surface of tissue

$$\frac{\partial \bar{P}'}{\partial r'} = 0 \text{ at } r' = R_2 + H'$$

There is no transfer of fluid through the annular ends of the tissue

$$\frac{\partial \bar{P}'}{\partial x'} = 0 \text{ at } x' = 0$$

$$\frac{\partial \bar{P}'}{\partial x'} = 0 \text{ at } x' = \ell'$$

Pressure across the boundary are assumed continuous

$$P' = \bar{P}' \text{ at } r' = R_2$$

At the entry of the capillary the fluid pressure is equal to the pressure at the arterial end we assume the fluid pressure equal to the pressure at venous end

$$P' = P_0' \text{ at } x' = 0$$

$$P' = P_\ell' \text{ at } x' = \ell'$$

IV. NON- DIMENSIONAL SCHEME

$$P = \frac{P'}{\rho u_{b_0}^2},\quad r = \frac{r'}{R_2},\quad u_0 = \frac{u_0'}{u_{b_0}},\quad u_c = \frac{u_c'}{u_{b_0}},$$

$$u_1 = \frac{u_1'}{u_{b_0}},\quad \bar{u} = \frac{\bar{u}'}{u_{b_0}}$$

$$x = \frac{x'}{R_2},\quad v_0 = \frac{v_0'}{u_{b_0}},\quad v_c = \frac{v_c'}{u_{b_0}},\quad v_1 = \frac{v_1'}{u_{b_0}},$$

$$\bar{v} = \frac{\bar{v}'}{u_{b_0}},\quad \bar{P} = \frac{\bar{P}'}{\rho u_{b_0}^2},$$

$$H = \frac{H'}{R_2},\quad \ell = \frac{\ell'}{R_2},\quad \tau_1 = \frac{\tau_1'}{\tau_0'}.$$

where ρ is the fluid density, u_{b_0} is the blood velocity and σ' is the slip parameter.

V. SOLUTION OF THE PROBLEM

Pressure in the tissue region, after solving Laplace equation, is given as

$$\bar{P} = E_0 + \frac{K\,Re}{R^2}\sum_{n=1}^{\infty} E_n \left\{ \frac{K_0\langle\alpha_n r\rangle I_1\langle\alpha_n\langle 1+H\rangle\rangle + I_0\langle\alpha_n r\rangle K_1\langle\alpha_n\langle 1+H\rangle\rangle}{I_1\langle\alpha_n\langle 1+H\rangle\rangle} \right\} \cos(\alpha_n x) \tag{10}$$

where $\alpha_n = \dfrac{n\pi}{\ell}$

Pressure in capillary region is obtained using equation of continuity, and given as

$$P = \frac{K}{2A\mu}\sum_{n=1}^{\infty} E_n \frac{1}{\alpha_n}\left\{ Y_n'(\alpha_n) + \frac{\alpha_n}{2} Y_n(\alpha_n) \right\} \cos(\alpha_n x) + c_1 x + c_2 \tag{11}$$

where

$$c_1 = \left\{ \frac{B}{2A}\left(\frac{B}{2A} + 1 \right) - A_1 \right\}$$

$$c_2 = P_0 - \frac{K}{2\mu A}\sum_{n=1}^{\infty} \frac{E_n}{\alpha_n}\left\{ Y_n'(\alpha_n) + \frac{\alpha_n}{2} Y_n(\alpha_n) \right\}$$

$$A_1 = -\frac{P_\ell - P_0}{\ell} - \frac{K}{2\mu A \ell}\sum_{n=1}^{\infty} \left\{ (-1)^n - 1 \right\} \left\{ Y_n'(\alpha_n) + \frac{\alpha_n}{2} Y_n(\alpha_n) \right\}$$

$$A = \frac{\rho u_{b_0} R_2}{\mu_0}\left\{ 1/16\mu_1 \left\{ \left({r_h}^3 - {R_1}^3 \right) + \left({R_1}^3 - 1 - 2\sigma \right) \right\} \right\}$$

$$A = \frac{1}{7}\left(\frac{\rho u_{b_0} R_2 \tau_0}{2\mu_0} \right)^{1/2} \left({r_h}^{5/2} - {R_1}^{5/2} \right)$$

$$E_0 = c_1 \ell$$

$$E_n = \frac{2c_1\left\{ (-1)^n - 1 \right\}}{\ell\alpha_n\left\{ Y_n(\alpha_n)\left(\frac{1}{2A} - 1 \right) + \frac{\alpha_n}{4A} Y_n'(\alpha_n) \right\}}$$

Flow rate in the capillary region is calculated as

$$Q_u = 2\pi\left\{ \int_0^{r_h} r\,u_c(0,r)\,dr + \int_{r_h}^{R_1} r\,u_1(0,r)\,dr + \int_{R_1}^{R_2} r\,u_0(0,r)\,dr \right\} \tag{12}$$

Introducing the velocities in the expression (12) we finally get the expression for flow rate as

$$Q_u = 2\pi\left[\frac{1}{\mu_1}\left\{ \frac{\rho u_{b_0} R_2}{4\mu_0}\frac{\partial P}{\partial x}\left(\frac{{r_h}^4 - {R_1}^4}{4} \right) + \tau_0\left(\frac{{r_h}^3 - {R_1}^3}{6} \right) - \frac{2}{7}\left(\frac{\rho u_{b_0} R_2}{2\mu_0}\tau_0 \right)^{1/2}\left({r_h}^{7/2} - {R_1}7/2 \right) \right\} + \frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x}\left(\frac{2 - {R_1}^4 - 4\sigma}{4} \right) \right] \tag{13}$$

where

$$u_0 = \frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x}\left(r^2 - 1 \right) - 2\sigma\frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x} + \frac{K}{\mu}\sum_{n=1}^{\infty} E_n \alpha_n Y_n(\alpha_n)\,\mathrm{Sin}(\alpha_n x)$$

$$u_1 = \frac{1}{\mu_1}\left[\frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x}\left(r^2 - {R_1}^2 \right) + \tau_0(r - R_1) - \frac{4}{3}\left(\frac{\rho u_{b_0} R_2}{\mu_0}\tau_0\frac{\partial P}{\partial x} \right)^{1/2}\left(r^{3/2} - {R_1}^{3/2} \right) \right] + \frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x}\left({R_1}^2 - 1 \right) - 2\sigma\frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x} + \frac{K}{\mu}\sum_{n=1}^{\infty} E_n \alpha_n Y_n(\alpha_n)\,\mathrm{Sin}(\alpha_n x)$$

$$u_c = \frac{1}{\mu_1}\left[\frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x}\left({r_h}^2 - {R_1}^2 \right) + \tau_0(r - R_1) - \frac{4}{3}\left(\frac{\rho u_{b_0} R_2}{\mu_0}\tau_0\frac{\partial P}{\partial x} \right)^{1/2}\left({r_h}^{3/2} - {R_1}^{3/2} \right) \right] + \frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x}\left({R_1}^2 - 1 \right) - 2\sigma\frac{\rho u_{b_0} R_2}{\mu_0}\frac{\partial P}{\partial x} + \frac{K}{\mu}\sum_{n=1}^{\infty} E_n \alpha_n Y_n(\alpha_n)\,\mathrm{Sin}(\alpha_n x)$$

The total volume of fluid transferred to the tissue per unit time is given by

$$Q_v = 2\pi R_2 \int_0^{\ell} v_0(x, R_2)\,dx \tag{14}$$

The quantity of fluid goes into the pores of the tissue is

$$\bar{Q}_v = 2\pi R_2 \int_0^{\ell/2} v_0(x, R_2 + H)\,dx \tag{15}$$

$$\bar{Q}_v = 2\pi R_2 \frac{K}{\mu}\sum_{n=1}^{\infty} E_n Y_n'(\alpha_n)\,\mathrm{Sin}\,\frac{n\pi}{2} \tag{16}$$

The ratio between $\bar{Q}_v$ and the total flow rate Q_v is characterized by filtration efficiency which is very important parameter of the system

$$\frac{\bar{Q}_v}{Q_u} = \frac{R_2}{F}\frac{K}{\mu}\sum_{n=1}^{\infty} E_n Y_n'(\alpha_n)\,\mathrm{Sin}\,\frac{n\pi}{2} \tag{17}$$

where

$$F=\left[\frac{1}{\mu}\left\{\frac{\rho u_{b_0}R_2}{4\mu_0}\frac{\partial P}{\partial x}\left(\frac{r_h{}^2-R_1{}^2}{4}\right)+\tau_0\left(\frac{r_h{}^3-R_1{}^3}{6}\right)-\frac{2}{7}\left(\frac{\rho u_{b_0}R_2}{\tau_0\mu_0}\frac{\partial P}{\partial x}\right)^{1/2}\left(r_h{}^{7/2}-R_1{}^{7/2}\right)\right\}\right.$$

$$\left.+\frac{\rho u_{b_0}R_2}{\mu_0}\frac{\partial P}{\partial x}\left(\frac{2-R_1{}^4-4\sigma}{4}\right)\right]$$

VI. RESULTS AND DISCUSSION

Results of the above analysis are presented through figures 2 to 11.

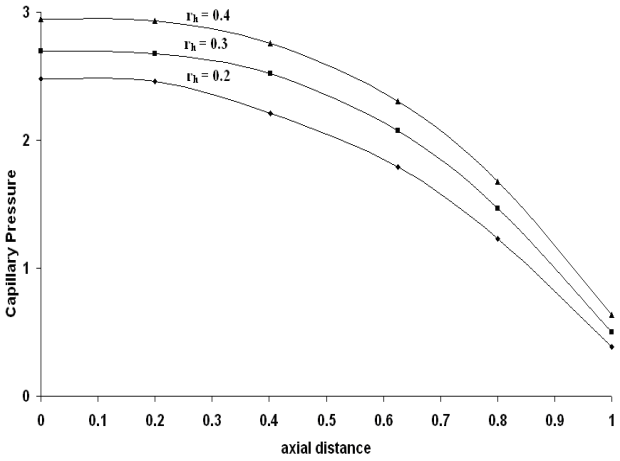

Fig.2 Variation of Pressure with axial distance for different core redius for a fixed value of dimensionless casson viscosity $\mu_1' = 0.1$

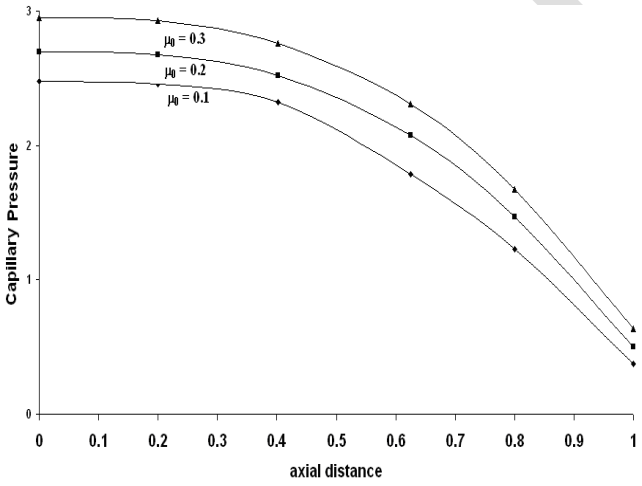

Fig.3 Variation of Pressure with axial distance for different values of peripheral layer viscosity casson viscosity $\mu_1' = 0.1$

Figures 2 and 3 shows the variation of capillary pressure with axial distance for different values of core thickness and casson viscosity respectively. Results of our model are similar to physiological fact. The pressure in the capillary region decreases towards the venous end which is clear from the graph of figure 2 and 3. The capillary pressure is also seem to depend on core radius of casson fluid model and peripheral layer viscosity. We may conclude from our observations that due to some disease if red cell starts depositing towards the axis of the capillary the pressure increases in the capillary region. Similar in diseased state when the viscosity of the base fluid i.e. plasma increase pressure in capillary region also rises.

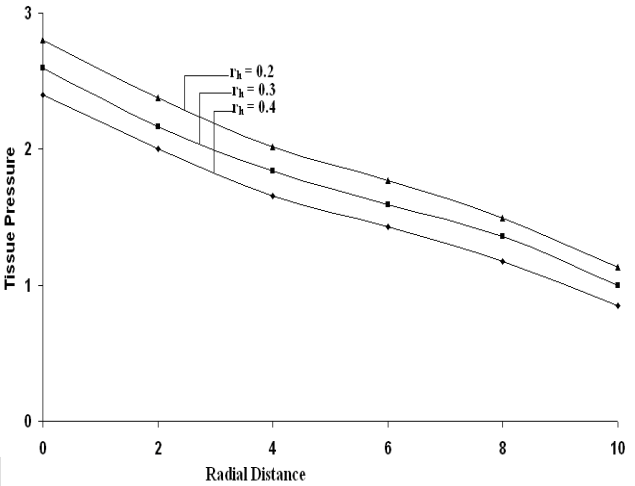

Fig. 4 Variation of tissue Pressure with radial distance for different values of core radius for fixed casson viscosity $\mu_1' = 0.1$

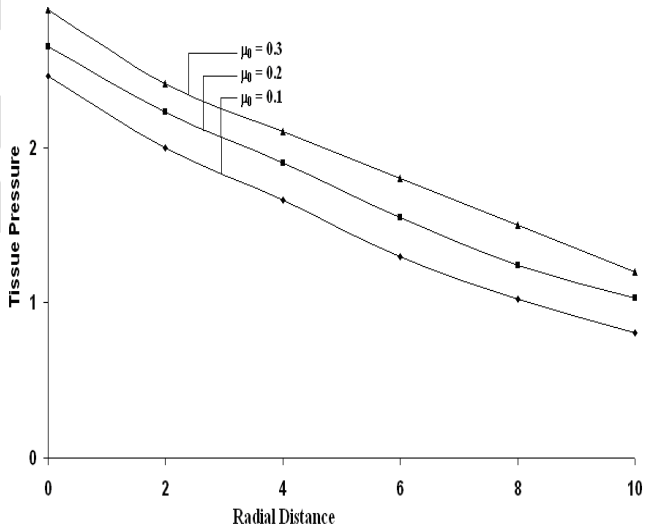

Fig.5 Variation of Tissue Pressure with radial distance for different values of core values of dimensionless peripheral layer viscosity

Fig. 4& 5 shows the variation of tissue pressure with radial distance for different values of core thickness and casson viscosity respectively. Tissue pressure decreases with radial distance i.e. towards as it moves for away from the capillary tissue interface. As core thickness increases tissue pressure decreases. Increase of core thickness represents the increased concentration of cells on the axis of the capillary. Tissue pressure increases as viscosity of cason fluid increases. Graphs also shown that pressure also decrease towards the venous end in tissue region.

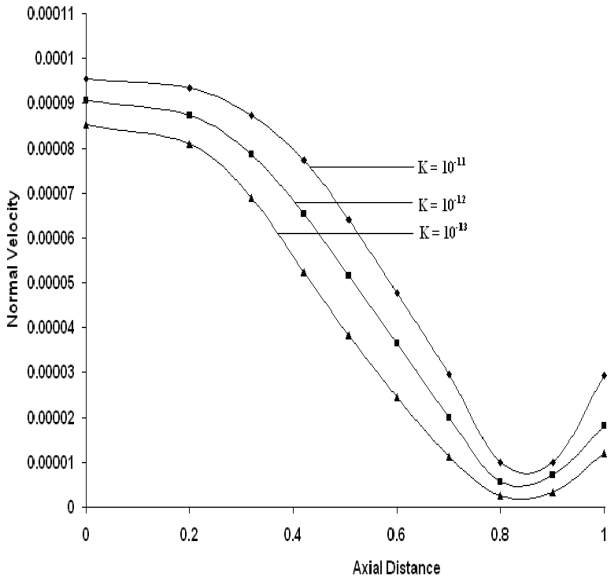

Fig. 6 Variation of Normal component of velocity with axial distance for different values of permeability

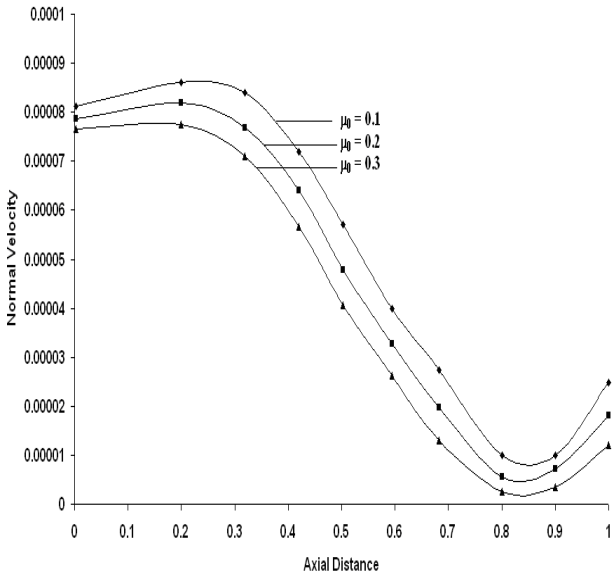

Fig. 7 Variation of Normal component of velocity with axial distance for different values of peripheral layer viscosity

Figure 6 and 7 show the variation of normal component of velocity at the interface for different values of permeability and peripheral layer viscosity. As fluid tends to the venous end the normal velocity falls. This is due to the frictional resistance of the capillary and the increased crossectional area. This depends on fact that the arterial pressure is higher than the venous pressure. Normal component of velocity increases as permeability increase and as viscosity of peripheral layer decreases the fluid becomes thin and normal velocity increases than fluid easily enters into the tissue.

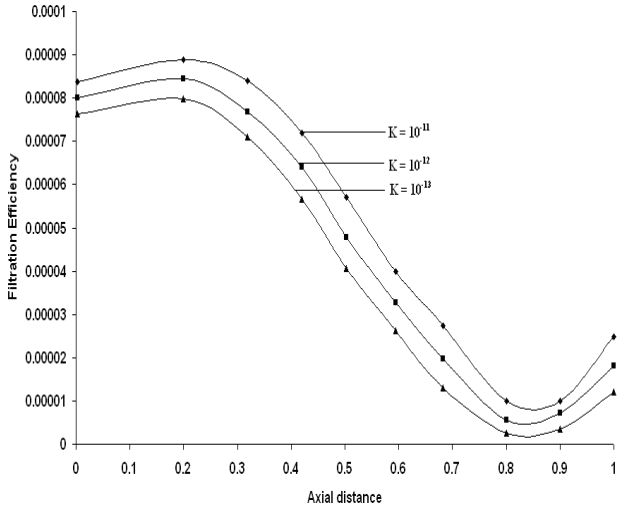

Figure 8 Variation of Filtration Efficiency with axial distance for different values of Permeability

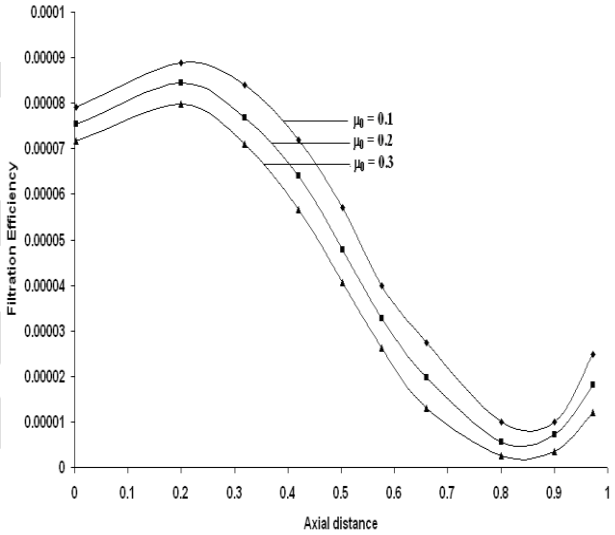

Figure 9 Variation of Filtration Efficiency with axial distance for different values of Peripheral layer viscosity

Figures 8 and 9 depict that the variation in filtration efficiency with respect to the axial distance for different values of permeability and peripheral layer viscosity. As axial distance increase first the filtration efficiency decrease then it increase slightly. This is due to the fact that fluids flow out of a capillary at the upstream end near an arteriole and reenter a capillary downstream near a venule. As permeability increases, filtration efficiency of fluid increases and more fluid enters into the tissue. As viscosity of peripheral layer increases the fluid becomes more thick which results that the less filtration of fluid to the tissue.

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