Sensitivity Analysis of Fractional Order Filter

Dr. Madhab Chandra Tripathy[#]

[#] *Department of Instrumentation and Electronics Engineering,*

College of engineering and technology, Bhubaneswar,

***Abstract*— In this paper the sensitivity analysis of continuous- time fractional order filters has been done to investigate 'how much the filter's behavior changes as a component value changes'. In particular, a KHN type fractional biquad filter has been taken where two fractional order elements of order α and β (0 < α, β ≤ 1) are used for realization. It is seen that pole frequency (ωn) and pole quality factor of fractional order filter is more sensitive towards component variations as compared to classical integer order filter; however transfer function sensitivities are less affected due to component variations.**

***Keywords*— KHN filter, pole frequency, Quality factor, Constant Phase Element, Biquad**

I. INTRODUCTION

Generally in signal processing, much of what is done in the analog domain is amplification and filtering [1]. Hence, there is a great importance for filter design in analog signal processing. However, because of component tolerance and opamp no idealities, the response of a practical filter is likely to deviate from the theoretical value. Even if some of the components are made adjustable, deviation will still arises because of component aging and thermal drift. It is therefore interest to know how sensitive a given filter is to component variation. Simply, sensitivity shows how much one thing/parameter changes as a function of another thing which is changing.

Mathematically, Sensitivity [i] is defined as

$$S_x^y = \lim_{\Delta x \to 0}[\frac{\Delta y / y}{\Delta x / x}] = \frac{x}{y}\frac{dy}{dx} \qquad \text{(i)}$$

Where x is the component that is varied and y is the filter characteristic (ωn or Q) in our case) that we wish to evaluate as x is varied.

Sensitivities are the exponents in the circuit equations. For example, Sensitivity of a pole frequency (ωn) w. r. t. capacitance c in a filter circuit is $S_c^{\omega_n} = 0.5$. This implies if capacitance increases by factor 4, then pole frequency (ωn) increases by factor $4^{0.5}$ i.e.2.

In this paper the sensitivity of the fractional order filter circuits have been investigated. Fractional calculus is an effective mathematical tool to describe the dynamics of fractional order filter circuit. Among the various applications of fractional calculus, modeling of real world phenomena, and physical systems, that is, voltage–current relationship in a nonideal capacitor [2], fractal behavior of metal-insulator-solution interface [3], biological systems [4-6] are few important ones. Recent research works show a trend in generalizing integer-order dynamics to fractional-order and study the performances through both simulation and experimentation [7]. Similar approach has been adopted to investigate the sensitivity of the fractional order filter.

This paper is organized as follows: Sect.2 presents the background of fractional order calculus. Sect.3 presents the basics of fractional order filter and their parameters. The sensitivity analysis of fractional order filter has been presented in Section 4 and concluding remarks are summarized in Sect.5.

II. BACKGROUND OF FRACTIONAL ORDER CALCULUS

Different definitions of fractional derivatives and fractional Integrals (Diffrintegrals) are considered. By means of them explicit formula and some special functions are derived. Fractional order derivative and integrals provide a powerful instrument for the description of memory and hereditary properties of different substances. This is the most significant advantage of fractional order models in comparison with integer models. Because of absence of appropriate mathematical models, fractional order systems were studied.

Fractional differential equations (also known as extraordinary differential equations) are a generalization of differential equations through the application of fractional calculus. The use of fractional calculus can improve and generalize well-established control methods and strategies.

There are two main approaches for defining a fractional derivative. The first considers differentiation and integration as limits of finite differences. The Grunwald-Letnikov definition follows this approach. The other approach generalizes a convolution type representation of repeated integration [5]. The Riemann-Liouville and Caputo definitions take this approach. Riemann-Liouville and Caputo fractional derivatives are fundamentally related to fractional integration operators. Consequently, the initial conditions of fractional derivatives are the frequency distributed and infinite dimensional state vector of fractional integrators.

Differential calculus is only the generalization of full integer order integral and differential calculus to real or even complex order. In the section below main definitions of fractional order integrals and derivatives are presented [8-10].

a. Riemann-Liouvill definition

b. Gr¨unwald-Letnikove definition

c. M. Caputo definition

Riemann–Liouville fractional integral

According to Riemann-Liouville the notion of fractional integral of Order α (α > 0) for a function f (t), is a natural consequence of the well known formula (Cauchy-Dirichlet, that reduces the calculation of the n-fold primitive of a function f (t) to a single integral of convolution type

$$\mathrm{a}D_t^{-\alpha}f(t) = \mathrm{a}I_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau \quad (1)$$

Riemann–Liouville fractional order derivative

The corresponding derivative is calculated using Lagrange's rule for differential operators. Computing n-th order derivative over the integral of order (n − α), the α order derivative is obtained. It is important to remark that n is the nearest integer bigger than α. [1]

$$aD_t^{\alpha}f(t) = \frac{d^n}{dt^n}aD_t^{-(n-\alpha)}f(t) = \frac{d^n}{dt^n}aI_t^{(n-\alpha)}f(t) \qquad (2)$$

Caputo fractional order calculus

There is another option for computing fractional derivatives; the Caputo fractional derivative. It was introduced by M. Caputo in his 1967 paper. In contrast to the Riemann Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows.

$$aD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}} \qquad (3)$$

Caputo’s definition of fractional order differential is given by:

$$\mathrm{L}[aD_t^{\alpha}f(t)] = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0) \qquad (4)$$

Grünwald–Letnikov derivative

In mathematics, the Grünwald–Letnikov derivative is a basic extension of the derivative in fractional calculus that allows one to take the derivative a non-integer number of times.

$$D^q f(x) = \lim_{h\to 0}\frac{\Delta_h^q f(x)}{h^q} \qquad (5)$$

Gr¨unwald-Letnikov definition of fractional order differ-integral:

$$D_\alpha^t = \lim_{h\to 0}\frac{1}{h}\sum_{j=0}^{\frac{1}{h}}(-1)^j\int_\propto^j f(t-jh) \qquad (6)$$

III. FRACTIONAL ORDER FILTER

The transfer function of a fractional order filter comprises of real powers of’s’ in numerator and denominator. This is in contrary to integer order filters where all the powers of’s’ in the transfer function are integers. The transfer functions of fractional order filters are represented by general fractional-order differential equations [11-12]. These filters are realized using either one or more fractional order elements [13,14].

It is well known that the fractional-order Kerwin Huelsman Newcomb (KHN) biquad filter provides a generalized filter structure with low pass, high pass and band pass characteristics at different terminals. Fractional order incorporation of this KHN filter has been taken up in this paper for sensitivity analysis as in Fig. 1 [17-19].

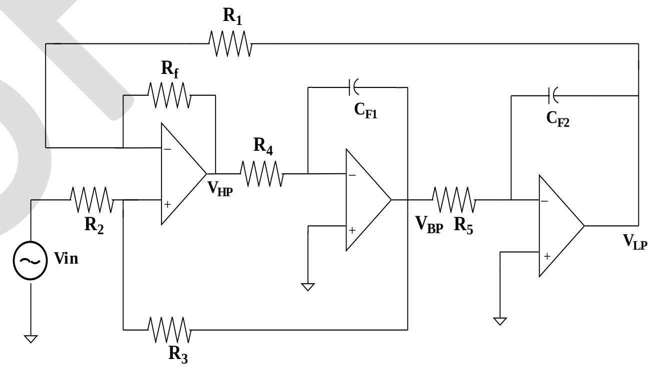

Fig.1 Fractional order KHN biquad Filter

Now, the transfer function of low pass, high pass and band pass responses can be expressed as

$$T_{lp} = \frac{\frac{2R_3}{R_2+R_3}}{R_4R_5C_{F1}C_{F2}S^{\alpha+\beta} + (\frac{2R_2}{R_2+R_3})R_5C_{F2}S^{\beta} + 1} \qquad (7)$$

$$T_{lp} = \frac{\frac{2R_3}{R_2+R_3}R_4R_5C_{F1}C_{F2}S^{\alpha+\beta}}{R_4R_5C_{F1}C_{F2}S^{\alpha+\beta} + (\frac{2R_2}{R_2+R_3})R_5C_{F2}S^{\beta} + 1} \qquad (8)$$

$$T_{lp}=\frac{(\frac{2R_3}{R_2+R_3})(\frac{2R_2}{R_2+R_3})R_5C_{F2}S^{\beta}}{R_4R_5C_{F1}C_{F2}S^{\alpha+\beta}+(\frac{2R_2}{R_2+R_3})R_5C_{F2}S^{\beta}+1} \qquad (9)$$

From the transfer function it is clear that the dc gain of the fractional order biquad filter is given as

$$K=\frac{2R_3}{R_2+R_3} \qquad (10)$$

However the other parameters like pole frequency (ωn), pole quality factor (Q) cannot be obtained directly. Assuming α=β, the characteristics equations of the above filter transfer function can be written as $s^{2\alpha}+as^{\alpha}+b$ where $b=\frac{1}{R_4R_5C_{F1}C_{F2}}$ and $a=(\frac{2R_2}{R_2+R_3})\frac{1}{R_4C_{F1}}$

Hence the ω_n and Q as in [4] can be expressed as

$$\omega_n=b^{\frac{1}{2\alpha}} \qquad (11)$$

$$Q=-\frac{1}{2\cos(\frac{\theta}{\alpha})} \qquad \theta=\cos^{-1}(\frac{-a}{2\sqrt{b}}) \qquad (12)$$

IV. SENSITIVITY ANALYSIS OF FRACTIONAL ORDER FILTER

A. *Gain sensitivity*

Applying equation (1) to the expression of gain (K) given in equation (5), we get

$$S_{R_3}^{K}=\frac{R_2}{(R_2+R_3)} \quad S_{R_2}^{K}=\frac{-R_2}{(R_2+R_3)} \qquad (13)$$

$$S_{R_1}^{K}=S_{R_4}^{K}=S_{R_5}^{K}=S_{C_{F1}}^{K}=S_{C_{F2}}^{K}=0$$

However, when $R_2=R_3$, then

$$S_{R_3}^{K}=-S_{R_2}^{K}=\frac{1}{2R_2} \qquad (14)$$

Since Gain sensitivities are inversely proportional to the value R2 (for R2=R3), a high practicable resistance should be used to reduce gain sensitivity. However K sensitivities is independent of other component variation. The gain sensitivity of a fractional order filter is same as classical integer order filter

B. *Pole frequency (ωn) sensitivity*:

Applying equation (1) to the expression of pole frequency (ωn) given in equation (6), we get

$$S_{R_4}^{\omega_0}=S_{R_5}^{\omega_0}=S_{C_{F1}}^{\omega_0}=S_{C_{F2}}^{\omega_0}=\frac{1}{2\alpha} \qquad (15)$$

$$S_{\alpha}^{\omega_0}=b^{\frac{1}{2\alpha}}\ln\sqrt{b} \qquad (16)$$

$$S_{R_2}^{\omega_0}=S_{R_3}^{\omega_0}=S_{R_1}^{\omega_0}=0$$

Since pole frequency sensitivities towards R4, R5, CF1 andCF2 are inversely proportional to the fractional order of FOE, a higher quality with lower tolerance factor FOE and resistance should be used when lower order filters are designed so that their variations towards component ageing and thermal drift will be minimum. However ωo sensitivity towards R2, R3, and R3 is zero which shows that these components variation does not have any impact on pole frequency. Similarly pole frequency sensitivities will reduce when FOE with lower exponent factor is used for realizing biquad filter.

It is seen that for a 2nd order biquad filter $(\alpha=1)$,

$$S_{R_4}^{\omega_0}=S_{R_5}^{\omega_0}=S_{C_{F1}}^{\omega_0}=S_{C_{F2}}^{\omega_0}=\frac{1}{2} \qquad (17)$$

$$S_{\alpha}^{\omega_0}=b^{\frac{1}{2}}\ln\sqrt{b}\,.$$

This indicates that pole frequency (ω_n) is more sensitivity towards component variations in fractional domain as α<1.

C. *Pole Quality factor (Q) sensitivity*

Applying equation (1) to the expression of quality factor (Q) given in equation (7), we get,

$$S_{R_1}^{\omega_0}=0,\; S_{\alpha}^{Q}=\frac{\theta}{\alpha}\tan(\frac{\theta}{\alpha}) \qquad (18)$$

$$S_{R_3}^{Q}=S_{R_4}^{Q}=-S_{R_2}^{Q}=-S_{R_5}^{Q}=S_{C_{F1}}^{Q}=-S_{C_{F2}}^{Q}=\frac{1}{2\alpha}\tan(\frac{\theta}{\alpha})\cot(\theta)$$

Similarly, for a 2nd order biquad filter $(\alpha=1)$

$$S_{R_3}^{Q}=S_{R_4}^{Q}=-S_{R_2}^{Q}=-S_{R_5}^{Q}=S_{C_{F1}}^{Q}=-S_{C_{F2}}^{Q}=\frac{1}{2}, \qquad (19)$$

This shows that quality factor (Q) sensitivity does not depend on the value of the components for a 2nd order biquad filter, however in case of fractional order filter Q sensitivity depends on the value of components.

D. *Transfer function sensitivity:-*

Transfer function sensitivities are more complex and thus difficult to put to use. Basically they are the functions of frequency and component values, which is shown in figure (1). Applying equation (1) to the expression of transfer function given in equation (2), we can derive the sensitivity of transfer function towards various components, as following. i.e. sensitivity towards α, β, R_2, R_3, R_4, R_5, C_{F1} and C_{F2} are given as

$$S_{\alpha}^{T_{LP}(s)} = \frac{\alpha \ln(s) s^{\alpha+\beta}}{s^{\alpha+\beta} + as^{\beta} + b} \tag{20}$$

$$S_{\beta}^{T_{LP}(s)} = \frac{\beta(s^{\alpha+\beta} + as^{\beta})\ln(s)}{s^{\alpha+\beta} + as^{\beta} + b} \tag{21}$$

$$S_{R_3}^{T_{LP}(s)} = -S_{R_2}^{T_{LP}(s)} = \frac{1}{2} + \frac{as^{\beta}}{2(s^{\alpha+\beta} + as^{\beta} + b)} \tag{22}$$

$$S_{R_4}^{T_{LP}(s)} = S_{C_{F1}}^{T_{LP}(s)} = -\frac{s^{\alpha+\beta}}{s^{\alpha+\beta} + as^{\beta} + b} \tag{23}$$

$$S_{R_5}^{T_{LP}(s)} = S_{C_{F2}}^{T_{LP}(s)} = -\frac{s^{\alpha+\beta} + as^{\beta}}{s^{\alpha+\beta} + as^{\beta} + b} \tag{24}$$

Now, the sensitivity functions for integer order filter can be obtained by putting α=1 and β=1 in above sensitivity functions. Since in case of fractional order filter, α and β are less than one, the transfer function sensitivity of integer order is more sensitive towards frequency as compared to fractional order filter.

V. CONCLUSION

This paper discusses the sensitivity analysis of continuous- time fractional order filters having two different exponents α and β (0 < α, β ≤ 1). Here, a KHN type fractional biquad filter has been taken where two fractional order elements of order α and β (0 < α, β ≤ 1) are used for realization. Various issues of parameter sensitivity of a KHN biquad filter have been elaborated. Simulation studies have also been carried out to explore the dependence of various design parameters on the order of fractional capacitors α and β. It has been observed that due to the presence of two parameter order α and β, additional degrees of design freedom are available for better flexibility of filter design [20,21]. It is seen that pole frequency (ωn) and pole quality factor of fractional order filter is more sensitive towards component variations as compared to classical integer order filter; however transfer function sensitivities are less affected due to component variations.

REFERENCES

[1]. S. Franco, "Design with operational amplifier and analog integrated circuits", TMH, third edition, 2002.
[2]. Westerlund S. Capacitor theory.IEEE Transaction on Dielectrics and Electrical Insulation1994;1(5):826–839.
[3]. Macdonald JR (ed.) Impedance Spectroscopy Emphasizing SolidMaterials and Systems. John Wiley and Sons: New York,1987.
[4]. Ionescu C, Muntean I, Machado JAT, Keyser RD, Abrudean M. A theoretical study on modeling the respiratory tract with ladder networks by means of intrinsic fractal geometry.IEEE Transactions on Biomedical Engineering2010; 57:246–253.
[5]. Magin RL.Fractional Calculus in Bioengineering. Begell House Publishers: Connecticut, 2006.
[6]. Moaddy K, Radwan AG, Salama KN, Momani S, Hashim I. The fractional-order modeling and synchronization of electrically coupled neuron systems.Computers & Mathematics with Applications2012;64(10):3329–3339.
[7]. Tripathy MC, Biswas K, Sen S. A design example of a fractional-order Kerwin-Huelsman-Newcomb (KHN) biquad filter with two fractional capacitors of different order. Circuits, Systems and Signal Processing2013;32(4):1523–1536.
[8]. S. Das, "Functional fractional calculus," Springer, 2011.
[9]. A. Loverro, "Fractional Calculus: History, Definitions and Applications for the Engineer," 2004.
[10]. K. B. Oldham and J. Spanier, "The Fractional Calculus," Academic Press. INC. 1974.
[11]. K. Biswas, S. Sen and P. Dutta, "Realization of a constant phase element and its performance study in a differentiator circuits", IEEE Trans.-II, vol. 53, pp. 802-806, 2006.
[12]. G. Radwan, A. S. Elwakil and A.M. Soliman, "On the realization of second-order filters to the fractional domain", Journal of circuits and system, vol. 18, No. 2, pp. 361-386, 2009.
[13]. A.G. Radwan, A. S. Elwakil and A.M. Soliman, "First order filter generalized to the fractional domain", J. circuits syst. Comput. , 17, pp. 55-66, 2008.
[14]. G. Carlson and C. halijak, "Approximation of fractional capacitors by regular Newton process", IEEE Trans. Circuits System, vol. 11, pp. 210-213, 1964.
[15]. Maundy, A. S. Elwakil, T. J. Freeborn, "On the practical realization of higher order filters with a fractional stepping", signal processing, vol. 91, pp. 484-491, 2011.
[16]. K.B. Oldham and C. G. Zoski, "Analogue instrumentation for processing polar graphic data" J.Electroanal Chem., vol. 157, pp. 27-51, 1983
[17]. I. Petras, "Stability of fractional-order System with rational order", ar Xiv: 0811.4102, vol. 2, [Math. Ds], 2008.
[18]. K. Biswas, S. Sen, and P. K. Dutta, 'Modeling of capacitive probe in a polarizable medium', Sensors and actuators A: Physical, vol. 120, pp. 115-122, 2005.
[19]. Dingyu Xue and yang Quan Chen, "Sub-optimum rational approximations to Fractional order linear system", Proceedings of IDETC/CIE, 2005.
[20]. A.G. Radwan, A.M. Soliman, A. S. Elwakil and A. Sedeek, "On the stability of linear system with fractional order elements, Science Direct, vol. 40, pp. 2317-2328, 2009.
[21]. D. Mondal, K. Biswas, Performance study of fractional order integrator using single component fractional order element. IET Circuits Devices Syst.5, 334–342 (2011).