Deteriorating Items Production Inventory Models with Two Warehouses under Shortages, Inflation and Permissible Delay in Payments

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Abstract: Production inventory model for deteriorating items with linear trend in demand, time varying holding cost, inflation and permissible delay in payments with two warehouses is developed. Shortages are allowed and completely backlogged. The excess units over the capacity of the own warehouse (OW) are stored in a rented warehouse (RW). Numerical examples are provided to illustrate the model.
Key Words: Inventory, Production, Two-warehouse, Deterioration, Inflation, Shortages, Permissible delay in payments

1. INTRODUCTION

Deteriorating items inventory models were widely studied in past. Ghare and Schrader [8] first developed an EOQ model with constant rate of deterioration. The model was extended by Covert and Philip [7] by considering variable rate of deterioration. Shah [21] further extended the model by considering shortages. The related work are found in (Nahmias [16], Raffat [19], Goyal and Giri [10], Wu et al. [23], Ouyang et al. [17]).

The economic order quantity model under the condition of permissible delay in payments was developed by Goyal [9]. The model was extended by Aggarwal and Jaggi [1] to consider the deteriorating items. Aggarwal and Jaggi's [1] model was further extended by Jamal et al. [12] to consider shortages. Teng et al. [22] developed an optimal pricing and lot sizing model by considering price sensitive demand under permissible delay in payments. A literature review on inventory model under trade credit is given by Chang et al. [5]. Min et al. [15] developed an inventory model for exponentially deteriorating items under conditions of permissible delay in payments.

In classical inventory models it is assumed that the available warehouse has unlimited capacity. But for taking advantage of price discounts, the retailer buys goods exceeding their own warehouse (OW) capacity. Therefore an additional storage facility may be needed to keep large stock. This additional storage space over the fixed capacity W of the own warehouse, may be a rented warehouse (RW) providing better preserving facility and charges higher rate for storage with a lower rate of deterioration. Hartley [11] first proposed a two warehouse inventory model. Sarma [20] developed an inventory model with finite rate of replenishment with two warehouses. Other research work related to two warehouse can be found in, for instance (Benkherouf [2], Bhunia and Maiti [3], Kar et al. [13], Chung and Huang [6]).

Liang and Zhou [14] considered a two warehouse inventory models for deteriorating items under conditionally permissible delay in payments. Yang [24] considered a two warehouse inventory problem for deteriorating items with constant rate of demand under inflation in two alternatives when shortages are completely backordered. Bhunia et al. [4] deals with a deterministic inventory model for linear trend in demand under inflationary conditions with different rates of deterioration in two warehouses. Patel and Parekh [18] studied two warehouse inventory model for deteriorating items under shortages, inflation and permissible delay in payments.

In this paper we have developed a two-warehouse production inventory model under time varying holding cost and linear demand with inflation and permissible delay in payments. Shortages are allowed and completely backlogged. Numerical examples are provided to illustrate the model and sensitivity analysis of the optimal solutions for major parameters is also carried out.

2. ASSUMPTIONS AND NOTATIONS

NOTATIONS:

The following notations are used for the development of the model:

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P(t) : Production rate is function of demand at time t, (kD(t), k>1)

D(t) : Demand rate is a linear function of time t (a+bt, a>0, 0<b<1)

A : Replenishment cost per order for two warehouse system

c : Purchasing cost per unit

p : Selling price per unit

c_2 : Shortage cost per unit

HC(OW): Holding cost per unit time is a linear function of time t (x_1+y_1t, $x_1>0$, $0<y_1<1$) in OW

HC(RW): Holding cost per unit time is a linear function of time t (x_2+y_2t, $x_2>0$, $0<y_2<1$) in RW

I_e : Interest earned per year

I_p : Interest charged per year

M : Permissible period of delay in settling the accounts with the supplier

T : Length of inventory cycle

I(t) : Inventory level at any instant of time t, $0 \le t \le T$

W : Capacity of owned warehouse

$I_1(t)$: Inventory level in OW at time t, $t \in [0,t_1]$

$I_2(t)$: Inventory level in RW at time t, $t \in [t_1, t_2]$

$I_3(t)$: Inventory level in RW at time t, $t \in [t_2, t_3]$

$I_4(t)$: Inventory level in OW at time t, $t \in [t_1, t_3]$

$I_5(t)$: Inventory level in OW at time t, $t \in [t_3, t_4]$

$I_6(t)$: Inventory level shortages in OW at time t, $t \in [t_4, t_5]$

$I_7(t)$: Inventory level in OW at time t, $t \in [t_5, T]$

Q_1 : Inventory level initially
Q_2 : Shortage of inventory
Q : Order quantity
R : Inflation rate
t_1 : Total time elapsed for storage of item in OW
t_2 : Production time
t_3 : Time to which inventory level becomes zero in RW
t_4 : Time to which inventory level becomes zero in OW
t_5 : Time to which shortage of inventory in OW
T : Cycle length
$\theta_1 t$: Deterioration rate in OW, $0< \theta_1<1$
$\theta_2 t$: Deterioration rate in RW, $0< \theta_2<1$
TC_i : Total relevant cost per unit time (i=1,2,3,4,5)

ASSUMPTIONS:

The following assumptions are considered for the development of two warehouse model.

- Production rate is a function of demand.
- The demand of the product is declining as a linear function of time.
- Replenishment rate is infinite and instantaneous.
- Lead time is zero.
- Shortages are allowed and completely backlogged.
- OW has a fixed capacity W units and the RW has unlimited capacity.
- The goods of OW are consumed only after consuming the goods kept in RW.
- The unit inventory costs per unit in the RW are higher than those in the OW.
- During the time, the account is not settled; generated sales revenue is deposited in an interest bearing account. At the end of the credit period, the account is settled as well as the buyer pays off all units sold and starts paying for the interest charges on the items in stocks.

3. THE MATHEMATICAL MODEL AND ANALYSIS

Production starts at time t=0 at the rate of P. The level of inventory increases to W up to time $t=t_1$ due to combined effect of production, demand and deterioration. Then inventory continues to stored in RW up to time $t=t_2$ till production stops. In the interval $[t_2,t_3]$ the inventory in RW gradually decreases due to demand and deterioration and it reaches to zero at $t=t_3$. In OW, however, the inventory W decreases during $[t_3, t_4]$ due to both demand and deterioration and by the time t_4, both warehouses are empty. Then shortages start up to time t_5 and then production starts at the rate of P. The figure describes the behaviour of inventory system.

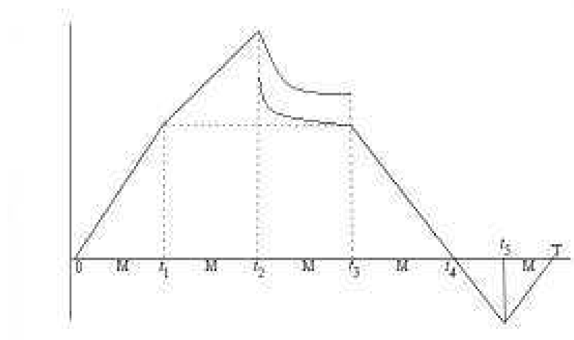


Figure 1

Hence, the inventory level at time t at RW and OW are governed by the following differential equations:

$$\frac{dI_1(t)}{dt} + \theta_1 t I_1(t) = (k-1)(a+bt), \qquad 0 \le t \le t_1 \qquad (1)$$

$$\frac{dI_2(t)}{dt} + \theta_2 t I_2(t) = (k-1)(a+bt), \qquad t_1 \le t \le t_2 \qquad (2)$$

$$\frac{dI_3(t)}{dt} + \theta_2 t I_3(t) = -(a+bt), \qquad t_2 \le t \le t_3 \qquad (3)$$

$$\frac{dI_4(t)}{dt} + \theta_1 t I_4(t) = 0, \qquad t_1 \le t \le t_3 \qquad (4)$$

$$\frac{dI_5(t)}{dt} + \theta_1 t I_5(t) = -(a+bt), \qquad t_3 \le t \le t_4 \qquad (5)$$

$$\frac{dI_6(t)}{dt} = -(a+bt), \qquad t_4 \le t \le t_5 \qquad (6)$$

$$\frac{dI_7(t)}{dt} = (k-1)(a+bt), \qquad t_5 \le t \le T \qquad (7)$$

with boundary conditions $I_1(0) = 0$, $I_2(t_1)=0$, $I_3(t_3)=0$, $I_4(t_1)=W$, $I_5(t_3)=W$, $I_6(t_4)=0$, $I_7(T)=0$

The solutions to equations (1) to (7) are given by:

$$I_1(t) = (k-1)\left[at + \frac{1}{2}bt^2 + \frac{1}{6}a\theta_1 t^3 + \frac{1}{8}b\theta_1 t^4 - \frac{1}{2}a\theta_1 t^3 - \frac{1}{4}b\theta_1 t^4\right], \qquad 0 \le t \le t_1 \quad (8)$$

$$I_2(t) = (k-1)\begin{bmatrix} a(t-t_1) + \frac{1}{2}b(t^2-t_1^2) + \frac{1}{6}a\theta_2(t^3-t_1^3) \\ +\frac{1}{8}b\theta_2(t^4-t_1^4) - \frac{1}{2}a\theta_2 t^2(t-t_1) - \frac{1}{4}b\theta_2 t^2(t^2-t_1^2) \end{bmatrix}, \qquad t_1 \le t \le t_2 \quad (9)$$

$$I_3(t) = \begin{bmatrix} a(t_3-t) + \frac{1}{2}b(t_3^2-t^2) + \frac{1}{6}a\theta_2(t_3^3-t^3) \\ +\frac{1}{8}b\theta_2(t_3^4-t^4) - \frac{1}{2}a\theta_2 t^2(t_3-t) - \frac{1}{4}b\theta_2 t^2(t_3^2-t^2) \end{bmatrix}, \qquad t_2 \le t \le t_3 \quad (10)$$

$$I_4(t) = W\left(1 + \frac{1}{2}\theta_1 t_1^2 - \frac{1}{2}\theta_1 t^2\right), \qquad t_1 \le t \le t_3 \quad (11)$$

$$I_5(t) = \begin{bmatrix} a(t_4 - t) + \frac{1}{2}b(t_4^2 - t^2) + \frac{1}{6}a\theta_2(t_4^3 - t^3) \\ + \frac{1}{8}b\theta_2(t_4^4 - t^4) - \frac{1}{2}a\theta_2 t^2(t_4 - t) - \frac{1}{4}b\theta_2 t^2(t_4^2 - t^2) \end{bmatrix} \quad t_3 \le t \le t_4 \quad (12)$$

$$I_6(t) = \left[-at - \frac{1}{2}bt^2 + at_4 + \frac{1}{2}bt_4^2 \right] \qquad t_4 \le t \le t_5 \quad (13)$$

$$I_7(t) = \left[k\left(at + \frac{1}{2}bt^2\right) - at - \frac{1}{2}bt^2 - kaT - \frac{1}{2}kbT^2 + aT + \frac{1}{2}bT^2 \right] \quad t_5 \le t \le T \quad (14)$$

(by neglecting higher powers of θ_1, θ_2)

Using the continuity of $I_2(t_2) = I_3(t_2)$ at $t = t_2$ in equations (9) and (10), we have

$$I_2(t_2) = (k-1)\begin{bmatrix} a(t_2 - t_1) + \frac{1}{2}b(t_2^2 - t_1^2) + \frac{1}{6}a\theta_2(t_2^3 - t_1^3) \\ + \frac{1}{8}b\theta_2(t_2^4 - t_1^4) - \frac{1}{2}a\theta_2 t_2^2(t_2 - t_1) - \frac{1}{4}b\theta_2 t_2^2(t_2^2 - t_1^2) \end{bmatrix}$$

$$= \begin{bmatrix} a(t_3 - t_2) + \frac{1}{2}b(t_3^2 - t_2^2) + \frac{1}{6}a\theta_2(t_3^3 - t_2^3) \\ + \frac{1}{8}b\theta_2(t_3^4 - t_2^4) - \frac{1}{2}a\theta_2 t_2^2(t_3 - t_2) - \frac{1}{4}b\theta_2 t_2^2(t_3^2 - t_2^2) \end{bmatrix} \quad (15)$$

which implies that

$$t_1 = \frac{-a(k-1) + \sqrt{\begin{matrix} a^2k^2 - 2a^2k + a^2 + 2abk^2t_2^2 + b^2k^2t_2^2 - b^2kt_3^2 \\ - 2abkt_3 - 2abkt_2 - b^2kt_2^2 + b^2t_3^2 + 2abt_3 \end{matrix}}}{b(k-1)} \quad (16)$$

(by neglecting higher powers of t_1, t_2 and t_3)

From equation (16), we note that t_1 is a function of t_2 and t_3, therefore t_1 is not a decision variable.

Similarly, Using the continuity of $I_4(t_3) = I_5(t_3)$ at $t = t_3$ in equations (11) and (12), we have

$$I_4(t_3) = W\left(1 + \frac{1}{2}\theta_1 t_1^2 - \frac{1}{2}\theta_1 t_3^2\right)$$

$$= \begin{bmatrix} a(T - t_3) + \frac{1}{2}b(t_4^2 - t_3^2) + \frac{1}{6}a\theta_1(t_4^3 - t_3^3) \\ + \frac{1}{8}b\theta_1(t_4^4 - t_3^4) - \frac{1}{2}a\theta_1 t_3^2(t_4 - t_3) - \frac{1}{4}b\theta_1 t_3^2(t_4^2 - t_3^2) \end{bmatrix} \quad (17)$$

$$t_4 = \frac{-a + \sqrt{a^2 + 2bW + bw\theta_1 t_1^2 - bw\theta_1 t_3^2 + b^2t_3^2 + 2abt_3}}{b} \quad (18)$$

(by neglecting higher powers of t_3 and t_4)

From equation (18), we note that t_4 is a function of t_1 and t_3, therefore t_4 is not a decision variable.

Similarly, Using the continuity of $I_6(t_5) = I_7(t_5)$ at $t = t_3$ in equations (13) and (14), we have

$$I_6(t_5) = \left[-at_5 - \frac{1}{2}bt_5^2 + at_4 + \frac{1}{2}bt_4^2 \right]$$

$$= \left[k\left(at_5 + \frac{1}{2}bt_5^2\right) - at_5 - \frac{1}{2}bt_5^2 - kaT - \frac{1}{2}kbT^2 + aT + \frac{1}{2}bT^2 \right] \quad (19)$$

which implies that

$$T = \frac{(kt_5 - t_4)}{(k-1)} \quad (20)$$

(by neglecting higher powers of t_5 and T)

From equation (20), we note that T is a function of t_4 and t_5, therefore T is not a decision variable.

Based on the assumptions and descriptions of the model, the total annual relevant costs TC_i, include the following elements:

(i) Ordering cost (OC) = A (21)

$$\text{(ii) HC(RW)} = \int_{t_1}^{t_2} (x_2+y_2t)I_2(t)e^{-Rt}dt + \int_{t_2}^{t_3} (x_2+y_2t)I_3(t)e^{-Rt}dt$$

$$= \int_{t_1}^{t_2} (x_2+y_2t)\left((k-1)\begin{bmatrix} a(t - t_1) + \frac{1}{2}b(t^2 - t_1^2) \\ + \frac{1}{6}a\theta_2(t^3 - t_1^3) + \frac{1}{8}b\theta_2(t^4 - t_1^4) \\ - \frac{1}{2}a\theta_2 t^2(t - t_1) - \frac{1}{4}b\theta_2 t^2(t^2 - t_1^2) \end{bmatrix} \right) e^{-Rt}dt$$

$$+ \int_{t_2}^{t_3} (x_2+y_2t)\left(\begin{bmatrix} a(t_3 - t) + \frac{1}{2}b(t_3^2 - t^2) \\ + \frac{1}{6}a\theta_2(t_3^3 - t^3) + \frac{1}{8}b\theta_2(t_3^4 - t^4) \\ -\frac{1}{2}a\theta_2 t^2(t_3 - t) - \frac{1}{4}b\theta_2 t^2(t_3^2 - t^2) \end{bmatrix} \right) e^{-Rt}dt$$

$$= x_2\left(at_3 + \frac{1}{2}bt_3^2 + \frac{1}{6}a\theta_2 t_3^3 + \frac{1}{8}b\theta_2 t_3^4\right)t_3 - \frac{1}{7}y_2R\left(\frac{1}{8}kb\theta_2 + \frac{1}{8}b\theta_2\right)t_2^7$$

$$+ \frac{1}{6}\left(\frac{1}{8}(y_2 - x_2R)b\theta_2 - \frac{1}{3}y_2Ra\theta_2\right)t_3^6$$

$$+ \frac{1}{5}\left(\frac{1}{8}x_2b\theta_2 + \frac{1}{3}(y_2 - x_2R)a\theta_2 - y_2R\left(-\frac{1}{2}b - \frac{1}{2}a\theta_2 t_3 - \frac{1}{4}b\theta_2 t_3^2\right)\right)t_3^5$$

$$+ \frac{1}{4}\left(\frac{1}{3}x_2a\theta_2 + (y_2 - x_2R)\left(-\frac{1}{2}b - \frac{1}{2}a\theta_2 t_3 - \frac{1}{4}b\theta_2 t_3^2\right) + y_2Ra\right)t_3^4$$

$$+ \frac{1}{3}\begin{pmatrix} x_2\left(-\frac{1}{2}b - \frac{1}{2}a\theta_2 t_3 - \frac{1}{4}b\theta_2 t_3^2\right) - (y_2 - x_2R)a \\ - y_2R\left(at_3 + \frac{1}{2}bt_3^2 + \frac{1}{6}a\theta_2 t_3^3 + \frac{1}{8}b\theta_2 t_3^4\right) \end{pmatrix} t_3^3$$

$$+ \frac{1}{2}\left(-x_2a + (y_2 - x_2R)\left(at_3 + \frac{1}{2}bt_3^2 + \frac{1}{6}a\theta_2 t_3^3 + \frac{1}{8}b\theta_2 t_3^4\right)\right)t_3^2$$

$$- \frac{1}{56}y_2Rb\theta_2 t_3^7 + \frac{1}{7}y_2R\left(-\frac{1}{8}kb\theta_2 + \frac{1}{8}b\theta_2\right)t_1^7$$

$$- x_2\left(at_3 + \frac{1}{2}bt_3^2 + \frac{1}{6}a\theta_2 t_3^3 + \frac{1}{8}b\theta_2 t_3^4\right)t_2 - \frac{1}{6}\left(\frac{1}{8}(y_2 - x_2R)b\theta_2 - \frac{1}{3}y_2Ra\theta_2\right)t_2^6$$

$$- \frac{1}{5}\left(\frac{1}{8}x_2 b\theta_2 + \frac{1}{3}(y_2-x_2R)a\theta_2 - y_2R\left(-\frac{1}{2}b - \frac{1}{2}a\theta_2 t_3 - \frac{1}{4}b\theta_2 t_3^2\right)\right)t_2^5$$

$$- \frac{1}{4}\left(\frac{1}{3}x_2 a\theta_2 + (y_2-x_2R)\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3 - \frac{1}{4}b\theta_2 t_3^2\right) + y_2Ra\right)t_2^4$$

$$-\frac{1}{3}\begin{pmatrix} x_2\left(-\frac12 b - \frac12 a\theta_2 t_3 - \frac14 b\theta_2 t_3^2\right) - (y_2-x_2R)a \\ - y_2R\left(at_3+\frac12 bt_3^2 + \frac16 a\theta_2 t_3^3+\frac18 b\theta_2 t_3^4\right)\end{pmatrix}t_2^3$$

$$-\frac12\left(-x_2a + (y_2-x_2R)\left(at_3+\frac12 bt_3^2+\frac16 a\theta_2t_3^3+\frac18 b\theta_2 t_3^4\right)\right)t_2^2$$

$$-x_2\left(k\left(-at_1-\frac12 bt_1^2-\frac16 a\theta_2 t_1^3 - \frac18 b\theta_2 t_1^4\right) + at_1+\frac12 bt_1^2+\frac16 a\theta_2 t_1^3+\frac18 b\theta_2 t_1^4\right)t_1$$

$$+\frac{1}{56}y_2Rb\theta_2 t_2^7 - \frac16\left((y_2-x_2R)\left(-\frac18 kb\theta_2+\frac18 b\theta_2\right) - y_2R\left(\frac13 a\theta_2 - \frac13 ka\theta_2\right)\right)t_1^6$$

$$-\frac15\begin{pmatrix} x_2\left(-\frac18 kb\theta_2+\frac18 b\theta_2\right) + (y_2-x_2R)\left(\frac13 a\theta_2 - \frac13 ka\theta_2\right) \\ -y_2R\left(-\frac12 a + k\left(\frac12 a + \frac12 a\theta_2 t_1 + \frac14 b\theta_2 t_1^2\right) - \frac14 b\theta_2 t_1^2 - \frac12 a\theta_2 t_1\right)\end{pmatrix}t_1^5$$

$$-\frac14\left(x_2\left(\frac13 a\theta_2 - \frac13 ka\theta_2\right) + (y_2-x_2R)\begin{pmatrix}-\frac12 b + k\left(\frac12 b+\frac12 a\theta_2 t_1+\frac14 b\theta_2 t_1^2\right)\\ -\frac14 b\theta_2 t_1^2 - \frac12 a\theta_2 t_1\end{pmatrix} - y_2R(-a+ka)\right)t_1^4$$

$$-\frac13\begin{pmatrix} x_2\begin{pmatrix}-\frac12 b + k\left(\frac12 b+\frac12 a\theta_2t_1+\frac14 b\theta_2 t_1^2\right)\\ -\frac14 b\theta_2 t_1^2-\frac12 a\theta_2 t_1\end{pmatrix} + (y_2-x_2R)(-a+ka) \\ -y_2R\left(k\left(-at_1-\frac12 bt_1^2-\frac16 a\theta_2t_1^3-\frac18 b\theta_2 t_1^4\right)+at_1+\frac12 bt_1^2+\frac16 a\theta_2 t_1^3+\frac18 b\theta_2 t_1^4\right)\end{pmatrix}t_1^3$$

$$-\frac12\left(x_2(-a+ka) + (y_2-x_2R)\begin{pmatrix}k\left(-at_1-\frac12 bt_1^2-\frac16 a\theta_2 t_1^3-\frac18 b\theta_2 t_1^4\right)\\ +at_1+\frac12 bt_1^2+\frac16 a\theta_2 t_1^3+\frac18 b\theta_2 t_1^4\end{pmatrix}\right)t_1^2$$

$$+x_2\left(k\left(-at_1-\frac12 bt_1^2-\frac16 a\theta_2t_1^3-\frac18 b\theta_2 t_1^4\right)+at_1+\frac12 bt_1^2+\frac16 a\theta_2t_1^3+\frac18 b\theta_2 t_1^4\right)t_2$$

$$+\frac16\left((y_2-x_2R)\left(-\frac18 kb\theta_2+\frac18 b\theta_2\right)-y_2R\left(\frac13 a\theta_2-\frac13 ka\theta_2\right)\right)t_2^6$$

$$+\frac15\begin{pmatrix}x_2\left(-\frac18 kb\theta_2+\frac18 b\theta_2\right)+(y_2-x_2R)\left(\frac13 a\theta_2-\frac13 ka\theta_2\right) \\ -y_2R\left(-\frac12 b + k\left(\frac12 b + \frac12 a\theta_2 t_1 + \frac14 b\theta_2 t_1^2\right) - \frac14 b\theta_2 t_1^2 - \frac12 a\theta_2 t_1\right)\end{pmatrix}t_2^5$$

$$+\frac14\begin{pmatrix}x_2\left(\frac13 a\theta_2-\frac13 ka\theta_2\right)+(y_2-x_2R)\begin{pmatrix}-\frac12 b + k\left(\frac12 b+\frac12 a\theta_2 t_1+\frac14 b\theta_2 t_1^2\right)\\ -\frac14 b\theta_2 t_1^2 - \frac12 a\theta_2 t_1\end{pmatrix}\\ -y_2R(-a+ka)\end{pmatrix}t_2^4$$

$$+\frac13\begin{pmatrix} x_2\begin{pmatrix}-\frac12 b + k\left(\frac12 b+\frac12 a\theta_2t_1+\frac14 b\theta_2 t_1^2\right)\\ -\frac14 b\theta_2 t_1^2-\frac12 a\theta_2 t_1\end{pmatrix} + (y_2-x_2R)(-a+ka)\\ -y_2R\begin{pmatrix}k\left(-at_1-\frac12 bt_1^2-\frac16 a\theta_2t_1^3-\frac18 b\theta_2 t_1^4\right)\\ +at_1+\frac12 bt_1^2+\frac16 a\theta_2 t_1^3+\frac18 b\theta_2 t_1^4\end{pmatrix}\end{pmatrix}t_2^3$$

$$+\frac12\left(x_2(-a+ka)+(y_2-x_2R)\begin{pmatrix}k\left(-at_1-\frac12 bt_1^2-\frac16 a\theta_2 t_1^3-\frac18 b\theta_2 t_1^4\right)\\ +at_1+\frac12 bt_1^2+\frac16 a\theta_2 t_1^3+\frac18 b\theta_2 t_1^4\end{pmatrix}\right)t_2^2 \qquad (22)$$

(by neglecting higher powers of R)

(iii) $$HC(OW) = \int_0^{t_1}(x_1+y_1t)I_1(t)e^{-Rt}dt + \int_{t_1}^{t_3}(x_1+y_1t)I_4(t)e^{-Rt}dt + \int_{t_3}^{t_4}(x_1+y_1t)I_5(t)e^{-Rt}dt$$

$$= \begin{pmatrix} -\frac17 y_1R\left(-\frac18 kb\theta_1+\frac18 b\theta_1\right)t_1^7 \\ +\frac16\left((y_1-x_1R)\left(-\frac18 kb\theta_1+\frac18 b\theta_1\right)-y_1R\left(-\frac13 ka\theta_1+\frac13 a\theta_1\right)\right)t_1^6 \\ +\frac15\begin{pmatrix}x_1\left(-\frac18 kb\theta_1+\frac18 b\theta_1\right)+(y_1-x_1R)\left(-\frac13 ka\theta_1+\frac13 a\theta_1\right)\\ -y_1R\left(\frac12 kb-\frac12 b\right)\end{pmatrix}t_1^5 \\ +\frac14\left(x_1\left(-\frac13 ka\theta_1+\frac13 a\theta_1\right)+(y_1-x_1R)\left(\frac12 kb-\frac12 b\right)-y_1R(ka-a)\right)t_1^4 \\ +\frac13\left(x_1\left(\frac12 kb-\frac12 b\right)+(y_1-x_1R)(ka-a)\right)t_1^3+\frac13 x_1(ka-a)t_1^2 \end{pmatrix}$$

$$+W\begin{pmatrix}\frac{1}{10}y_1R\theta_1 t_3^5 - \frac18(y_1-x_1R)\theta_1 t_3^4 + \frac13\left(-\frac12 x_1\theta_1 - y_1R\left(1+\frac12\theta_1 t_1^2\right)\right)t_3^3 \\ +\frac12(y_1-x_1R)\left(1+\frac12\theta_1 t_1^2\right)t_3^2 + x_1\left(1+\frac12\theta_1 t_1^2\right)t_3\end{pmatrix}$$

$$-W\begin{pmatrix}\frac{1}{10}y_1R\theta_1 t_1^5 - \frac18(y_1-x_1R)\theta_1 t_1^4 + \frac13\left(-\frac12 x_1\theta_1 - y_1R\left(1+\frac12\theta_1 t_1^2\right)\right)t_1^3 \\ +\frac12(y_1-x_1R)\left(1+\frac12\theta_1 t_1^2\right)t_1^2 + x_1\left(1+\frac12\theta_1 t_1^2\right)t_1\end{pmatrix}$$

$$+\begin{pmatrix} -\frac{1}{56}y_1Rb\theta_1 t_4^7 + \frac16\left(\frac18(y_1-x_1R)b\theta_1 - \frac13 y_1Ra\theta_1\right)t_4^6 \\ +\frac15\left(\frac18 x_1b\theta_1 + \frac13(y_1-x_1R)a\theta_1 - y_1R\left(-\frac12\theta_1\left(\frac12 bT^2+at_4\right)-\frac12 b\right)\right)t_4^5 \\ +\frac14\left(\frac13 x_1a\theta_1 + (y_1-x_1R)\left(-\frac12\theta_1\left(\frac12 bt_4^2+at_4\right)-\frac12 b\right)+y_1Ra\right)t_4^4 \\ +\frac13\begin{pmatrix}x_1\left(-\frac12\theta_1\left(\frac12 bt_4^2+at_4\right)-\frac12 b\right)\\ -(y_1-x_1R)a - y_1R\left(\frac18 b\theta_1 t_4^4+\frac16 a\theta_1 t_4^3+\frac12 bt_4^2+at_4\right)\end{pmatrix}t_4^3\end{pmatrix}$$

$$+\begin{pmatrix} +\frac12\left(-x_1a - (y_1-x_1R)\left(\frac18 b\theta_1 t_4^4+\frac16 a\theta_1 t_4^3+\frac12 bt_4^2+at_4\right)\right)t_4^2 \\ +x_1\left(\frac18 b\theta_1 t_4^4+\frac16 a\theta_1 t_4^3+\frac12 bt_4^2+at_4\right)t_4\end{pmatrix}$$

$$+\begin{pmatrix}\frac{1}{56}y_1Rb\theta_1 t_3^7 - \frac16\left(\frac18(y_1-x_1R)b\theta_1 - \frac13 y_1Ra\theta_1\right)t_3^6 \\ -\frac15\left(\frac18 x_1b\theta_1 + \frac13(y_1-x_1R)a\theta_1 - y_1R\left(-\frac12\theta_1\left(\frac12 bt_4^2+at_4\right)-\frac12 b\right)\right)t_3^5 \\ -\frac14\left(\frac13 x_1a\theta_1 + (y_1-x_1R)\left(-\frac12\theta_1\left(\frac12 bt_4^2+at_4\right)-\frac12 b\right)+y_1Ra\right)t_3^4\end{pmatrix}$$

$$
+\left(\begin{array}{l}
-\dfrac{1}{3}\left(\begin{array}{l} x_1\left(-\dfrac{1}{2}\theta_1\left(\dfrac{1}{2}bt_4^2+at_4\right)-\dfrac{1}{2}b\right) \\ -\left(y_1-x_1R\right)a+y_1R\left(\dfrac{1}{8}b\theta_1t_4^4+\dfrac{1}{6}a\theta_1t_4^3+\dfrac{1}{2}bt_4^2+at_4\right)\end{array}\right)t_3^3 \\
-\dfrac{1}{2}\left(-x_1a-\left(y_1-x_1R\right)\left(\dfrac{1}{8}b\theta_1t_4^4+\dfrac{1}{6}a\theta_1t_4^3+\dfrac{1}{2}bt_4^2+at_4\right)\right)t_3^2 \\
-x_1\left(\dfrac{1}{8}b\theta_1t_4^4+\dfrac{1}{6}a\theta_1t_4^3+\dfrac{1}{2}bt_4^2+at_4\right)t_3
\end{array}\right) \qquad (23)
$$

(iv) Deterioration cost:

$$
DC = c\left[\begin{array}{l}\displaystyle\int_{t_1}^{t_2}\theta_2 tI_2(t)e^{-Rt}dt+\int_{t_2}^{t_3}\theta_2 tI_3(t)e^{-Rt}dt \\ \displaystyle +\int_{0}^{t_1}\theta_1 tI_1(t)e^{-Rt}dt+\int_{t_1}^{t_3}\theta_1 tI_4(t)e^{-Rt}dt+\int_{t_3}^{t_4}\theta_1 tI_5(t)e^{-Rt}dt\end{array}\right]
$$

$$
= c\theta_2\left[\begin{array}{l}
-\dfrac{1}{7}R\left(-\dfrac{1}{8}kb\theta_2+\dfrac{1}{8}b\theta_2\right)t_2^7+\dfrac{1}{6}\left(\begin{array}{l}-\dfrac{1}{8}kb\theta_2+\dfrac{1}{8}b\theta_2 \\ -R\left(\dfrac{1}{3}a\theta_2-\dfrac{1}{3}ka\theta_2\right)\end{array}\right)t_2^6 \\
+\dfrac{1}{5}\left(\dfrac{1}{3}a\theta_2-\dfrac{1}{3}ka\theta_2-R\left(\begin{array}{l}-\dfrac{1}{2}b+k\left(\dfrac{1}{2}b+\dfrac{1}{2}a\theta_2t_1+\dfrac{1}{4}b\theta_2t_1^2\right) \\ -\dfrac{1}{4}b\theta_2t_1^2-\dfrac{1}{2}a\theta_2t_1\end{array}\right)\right)t_2^5 \\
+\dfrac{1}{4}\left(\begin{array}{l}-\dfrac{1}{2}b+k\left(\dfrac{1}{2}b+\dfrac{1}{2}a\theta_2t_1+\dfrac{1}{4}b\theta_2t_1^2\right)-\dfrac{1}{4}b\theta_2t_1^2 \\ -\dfrac{1}{2}a\theta_2t_1-R\left(-a+ka\right)\end{array}\right)t_2^4 \\
+\dfrac{1}{3}\left(-a+ka-R\left(\begin{array}{l}k\left(-at_1-\dfrac{1}{2}bt_1^2-\dfrac{1}{6}a\theta_2t_1^3-\dfrac{1}{8}b\theta_2t_1^4\right) \\ +at_1+\dfrac{1}{2}bt_1^2+\dfrac{1}{6}a\theta_2t_1^3+\dfrac{1}{8}b\theta_2t_1^4\end{array}\right)\right)t_2^3 \\
+\dfrac{1}{2}\left(k\left(-at_1-\dfrac{1}{2}bt_1^2-\dfrac{1}{6}a\theta_2t_1^3-\dfrac{1}{8}b\theta_2t_1^4\right)+at_1+\dfrac{1}{2}bt_1^2+\dfrac{1}{6}a\theta_2t_1^3+\dfrac{1}{8}b\theta_2t_1^4\right)t_2^2
\end{array}\right]
$$

$$
-c\theta_2\left[\begin{array}{l}
-\dfrac{1}{7}R\left(-\dfrac{1}{8}kb\theta_2+\dfrac{1}{8}b\theta_2\right)t_1^7+\dfrac{1}{6}\left(\begin{array}{l}-\dfrac{1}{8}kb\theta_2+\dfrac{1}{8}b\theta_2 \\ -R\left(\dfrac{1}{3}a\theta_2-\dfrac{1}{3}ka\theta_2\right)\end{array}\right)t_1^6 \\
\dfrac{1}{5}\left(\dfrac{1}{3}a\theta_2-\dfrac{1}{3}ka\theta_2-R\left(\begin{array}{l}-\dfrac{1}{2}b+k\left(\dfrac{1}{2}b+\dfrac{1}{2}a\theta_2t_1+\dfrac{1}{4}b\theta_2t_1^2\right) \\ -\dfrac{1}{4}b\theta_2t_1^2-\dfrac{1}{2}a\theta_2t_1\end{array}\right)\right)t_1^5 \\
+\dfrac{1}{4}\left(\begin{array}{l}-\dfrac{1}{2}b+k\left(\dfrac{1}{2}b+\dfrac{1}{2}a\theta_2t_1+\dfrac{1}{4}b\theta_2t_1^2\right) \\ -\dfrac{1}{4}b\theta_2t_1^2-\dfrac{1}{2}a\theta_2t_1-R\left(-a+ka\right)\end{array}\right)t_1^4
\end{array}\right]
$$

$$
-c\theta_2\left[\begin{array}{l}
\dfrac{1}{3}\left(-a+ka-R\left(\begin{array}{l}k\left(-at_1-\dfrac{1}{2}bt_1^2-\dfrac{1}{6}a\theta_2t_1^3-\dfrac{1}{8}b\theta_2t_1^4\right) \\ +at_1+\dfrac{1}{2}bt_1^2+\dfrac{1}{6}a\theta_2t_1^3+\dfrac{1}{8}b\theta_2t_1^4\end{array}\right)\right)t_1^3 \\
+\dfrac{1}{2}\left(\begin{array}{l}k\left(-at_1-\dfrac{1}{2}bt_1^2-\dfrac{1}{6}a\theta_2t_1^3-\dfrac{1}{8}b\theta_2t_1^4\right) \\ +at_1+\dfrac{1}{2}bt_1^2+\dfrac{1}{6}a\theta_2t_1^3+\dfrac{1}{8}b\theta_2t_1^4\end{array}\right)t_1^2
\end{array}\right]
$$

$$
+c\theta_2\left[\begin{array}{l}
-\dfrac{1}{56}Rb\theta_2t_3^7+\dfrac{1}{6}\left(\dfrac{1}{8}b\theta_2-\dfrac{1}{3}Ra\theta_2\right)t_3^6 \\
+\dfrac{1}{5}\left(\dfrac{1}{3}a\theta_2-R\left(-\dfrac{1}{2}b-\dfrac{1}{2}a\theta_2t_3-\dfrac{1}{4}b\theta_2t_3^2\right)\right)t_3^5 \\
+\dfrac{1}{4}\left(-\dfrac{1}{2}b-\dfrac{1}{2}a\theta_2t_3-\dfrac{1}{4}b\theta_2t_3^2+Ra\right)t_3^4 \\
+\dfrac{1}{3}\left(-a-R\left(at_3+\dfrac{1}{2}bt_3^2+\dfrac{1}{6}a\theta_2t_3^3+\dfrac{1}{8}b\theta_2t_3^4\right)\right)t_3^3 \\
+\dfrac{1}{2}\left(at_3+\dfrac{1}{2}bt_3^2+\dfrac{1}{6}a\theta_2t_3^3+\dfrac{1}{8}b\theta_2t_3^4\right)t_3^2
\end{array}\right]
$$

$$
-c\theta_2\left[\begin{array}{l}
-\dfrac{1}{56}Rb\theta_2t_2^7+\dfrac{1}{6}\left(\dfrac{1}{8}b\theta_2-\dfrac{1}{3}Ra\theta_2\right)t_2^6 \\
+\dfrac{1}{5}\left(\dfrac{1}{3}a\theta_2-R\left(-\dfrac{1}{2}b-\dfrac{1}{2}a\theta_2t_3-\dfrac{1}{4}b\theta_2t_3^2\right)\right)t_2^5 \\
+\dfrac{1}{4}\left(-\dfrac{1}{2}b-\dfrac{1}{2}a\theta_2t_3-\dfrac{1}{4}b\theta_2t_3^2+Ra\right)t_2^4 \\
+\dfrac{1}{3}\left(-a-R\left(at_3+\dfrac{1}{2}bt_3^2+\dfrac{1}{6}a\theta_2t_3^3+\dfrac{1}{8}b\theta_2t_3^4\right)\right)t_2^3 \\
+\dfrac{1}{2}\left(at_3+\dfrac{1}{2}bt_3^2+\dfrac{1}{6}a\theta_2t_3^3+\dfrac{1}{8}b\theta_2t_3^4\right)t_2^2
\end{array}\right]
$$

$$
+c\theta_1\left[\begin{array}{l}
-\dfrac{1}{7}R\left(-\dfrac{1}{8}kb\theta_1+\dfrac{1}{8}b\theta_1\right)t_1^7 \\
+\dfrac{1}{6}\left(-\dfrac{1}{8}kb\theta_1+\dfrac{1}{8}b\theta_1-R\left(-\dfrac{1}{3}ka\theta_1+\dfrac{1}{3}a\theta_1\right)\right)t_1^6 \\
+\dfrac{1}{5}\left(-\dfrac{1}{3}ka\theta_1+\dfrac{1}{3}a\theta_1-R\left(\dfrac{1}{2}kb-\dfrac{1}{2}b\right)\right)t_1^5 \\
+\dfrac{1}{4}\left(\dfrac{1}{2}kb-\dfrac{1}{2}b-R\left(-a+ka\right)\right)t_1^4+\dfrac{1}{3}\left(-a+ka\right)t_1^3
\end{array}\right]
$$

$$
+c\theta_1\left[\begin{array}{l}
-\dfrac{1}{7}R\left(-\dfrac{1}{8}kb\theta_1+\dfrac{1}{8}b\theta_1\right)t_1^7 \\
+\dfrac{1}{6}\left(-\dfrac{1}{8}kb\theta_1+\dfrac{1}{8}b\theta_1-R\left(-\dfrac{1}{3}ka\theta_1+\dfrac{1}{3}a\theta_1\right)\right)t_1^6 \\
+\dfrac{1}{5}\left(-\dfrac{1}{3}ka\theta_1+\dfrac{1}{3}a\theta_1-R\left(\dfrac{1}{2}kb-\dfrac{1}{2}b\right)\right)t_1^5 \\
+\dfrac{1}{4}\left(\dfrac{1}{2}kb-\dfrac{1}{2}b-R\left(-a+ka\right)\right)t_1^4+\dfrac{1}{3}\left(-a+ka\right)t_1^3
\end{array}\right]
$$

$$+c\theta_1 W\left[\frac{1}{10}R\theta_1 t_3^5-\frac{1}{8}\theta_1 t_3^4-\frac{1}{3}R\left(1+\frac{1}{2}\theta_1 t_1^2\right)t_3^3+\frac{1}{2}\left(1+\frac{1}{2}\theta_1 t_1^2\right)t_3^2\right]$$

$$-c\theta_1 W\left[\frac{1}{10}R\theta_1 t_1^5-\frac{1}{8}\theta_1 t_1^4+\frac{1}{3}R\left(1+\frac{1}{2}\theta_1 t_1^2\right)t_1^3+\frac{1}{2}\left(1+\frac{1}{2}\theta_1 t_1^2\right)t_1^2\right]$$

$$+c\theta_1\left[\begin{array}{l}-\frac{1}{56}R\theta_1 bt_4^7+\frac{1}{6}\left(\frac{1}{8}b\theta_1-\frac{1}{3}Ra\theta_1\right)t_4^6\\+\frac{1}{5}\left(\frac{1}{3}a\theta_1-R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_4^5\\+\frac{1}{4}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_4^4\\+\frac{1}{3}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_4^3\\+\frac{1}{2}\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_4^2\end{array}\right]$$

$$-c\theta_1\left[\begin{array}{l}-\frac{1}{56}R\theta_1 bt_3^7+\frac{1}{6}\left(\frac{1}{8}b\theta_1-\frac{1}{3}Ra\theta_1\right)t_3^6\\+\frac{1}{5}\left(\frac{1}{3}a\theta_1-R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_3^5\\+\frac{1}{4}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_3^4\\+\frac{1}{3}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_3^3\\+\frac{1}{2}\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_3^2\end{array}\right] \quad (24)$$

(v) Shortage cost is given by

$$SC=-c_2\int_{t_4}^{t_5}I_6(t)e^{-Rt}dt-c_2\int_{t_5}^{T}I_7(t)e^{-Rt}dt$$

$$=-c_2\int_{t_4}^{t_5}\left(-at-\frac{1}{2}bt^2+at_4+\frac{1}{2}bt_4^2\right)e^{-Rt}dt$$

$$-c_2\int_{t_5}^{T}\left(\begin{array}{l}k\left(at+\frac{1}{2}bt^2\right)-at-\frac{1}{2}bt^2-kaT\\-\frac{1}{2}kbT^2+aT+\frac{1}{2}bT^2\end{array}\right)e^{-Rt}dt$$

$$=-\frac{c_2}{8}\left(\begin{array}{l}Rkbt_5^4+\frac{8}{5}\left(aR-\frac{1}{2}b\right)kt_5^3+(-2Rb(k-1)T^2\\-4aR(k-1)T+\left(-4at_4-2bt_4^2\right)R-4ka)t_5^2\\+\left(4b(k-1)T^2+8a(k-1)T+8at_4+4bt_4^2\right)t_5\\+Rb(k-1)T^4+\frac{4}{3}(k-1)\left(-2b+aR\right)T^3\\-4a(k-1)T^2+\frac{4}{3}t_4^2\left(\left(\frac{3}{4}t_4+a\right)t_4R-3a-2bt_4\right)\end{array}\right) \quad (25)$$

(vi) Interest Earned: There are two cases:

Case I : $0 \le M \le t_4$:

In this case interest earned is:

$$IE_1=pI_e\int_0^M\left(a+bt\right)te^{-Rt}dt$$

$$=pI_e\left[-\frac{1}{4}bRM^4+\frac{1}{3}\left(-Ra+b\right)M^3+\frac{1}{2}aM^2\right] \quad (26)$$

Case II : $t_4 \le M \le T$:

In this case interest earned is:

$$IE_2=pI_e\left(\int_0^{t_4}\left(a+bt\right)te^{-Rt}dt+\left(a+bt_4\right)t_4\left(M-t_4\right)\right)$$

$$=pI_e\left[\begin{array}{l}-\frac{1}{4}bRt_4^4+\frac{1}{3}\left(-Ra+b\right)t_4^3\\+\frac{1}{2}at_4^2+\left(a+bt_0\right)t_4\left(M-t_4\right)\end{array}\right] \quad (27)$$

(vii) Interest Payable: There are five cases described as in figure:

Case I : $0 \le M \le t_1$:

In this case, annual interest payable is:

$$IP_1=cI_p\left[\begin{array}{l}\int_M^{t_1}I_1(t)e^{-Rt}dt+\int_{t_1}^{t_2}I_2(t)e^{-Rt}dt+\int_{t_2}^{t_3}I_3(t)e^{-Rt}dt\\+\int_{t_1}^{t_3}I_4(t)e^{-Rt}dt+\int_{t_3}^{t_4}I_5(t)e^{-Rt}dt\end{array}\right]$$

$$=cI_p\left[\begin{array}{l}-\frac{1}{48}R\theta_2 bt_3^6+\frac{1}{5}\left(\frac{1}{8}\theta_2 b-\frac{1}{3}R\theta_2 a\right)t_3^5\\+\frac{1}{4}\left(\frac{1}{3}\theta_2 a-R\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2\right)\right)t_3^4\\+\frac{1}{3}\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2+Ra\right)t_3^3\\+\frac{1}{2}\left(-a-R\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2 t_3^3+\frac{1}{8}b\theta_2 t_3^4\right)\right)t_3^2\\+at_3^2+\frac{1}{2}bt_3^3+\frac{1}{6}a\theta_2 t_3^4+\frac{1}{8}b\theta_2 t_3^5\end{array}\right]$$

$$-cI_p\left[\begin{array}{l}-\frac{1}{48}R\theta_2 bt_2^6+\frac{1}{5}\left(\frac{1}{8}\theta_2 b-\frac{1}{3}R\theta_2 a\right)t_2^5\\+\frac{1}{4}\left(\frac{1}{3}\theta_2 a-R\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2\right)\right)t_2^4\\+\frac{1}{3}\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2+Ra\right)t_2^3\\+\frac{1}{2}\left(-a-R\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2 t_3^3+\frac{1}{8}b\theta_2 t_3^4\right)\right)t_2^2\\+\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2 t_3^3+\frac{1}{8}b\theta_2 t_3^4\right)t_2\end{array}\right]$$

$$+cI_pW\left[\frac{1}{8}R\theta_1t_3^4-\frac{1}{6}\theta_1t_3^3-\frac{1}{2}R\left(1+\frac{1}{2}\theta_1t_1^2\right)t_3^2+t_3+\frac{1}{2}\theta_1t_1^2t_3\right]$$

$$-cI_pW\left[\frac{1}{8}R\theta_1t_1^4-\frac{1}{6}\theta_1t_1^3-\frac{1}{2}R\left(1+\frac{1}{2}\theta_1t_1^2\right)t_1^2+t_1\right]$$

$$+cI_p\left[\begin{array}{l}-\frac{1}{48}R\theta_1bt_4^6+\frac{1}{5}\left(\frac{1}{8}\theta_1b-\frac{1}{3}R\theta_1a\right)t_4^5\\ +\frac{1}{4}\left(\frac{1}{3}\theta_1a-R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_4^4\\ +\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_4^3\\ +\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1t_4^4+\frac{1}{6}a\theta_1t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_4^2\\ +\left(\frac{1}{8}b\theta_1t_4^4+\frac{1}{6}a\theta_1t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_4\end{array}\right]$$

$$-cI_p\left[\begin{array}{l}-\frac{1}{48}R\theta_1bt_3^6+\frac{1}{5}\left(\frac{1}{8}\theta_1b-\frac{1}{3}R\theta_1a\right)t_3^5\\ +\frac{1}{4}\left(\frac{1}{3}\theta_1a-R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_3^4\\ +\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_3^3\\ +\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1t_4^4+\frac{1}{6}a\theta_1t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_3^2\\ +\left(\frac{1}{8}b\theta_1t_4^4+\frac{1}{6}a\theta_1t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_3\end{array}\right]$$

$$+cI_p\left[\begin{array}{l}-\frac{1}{6}R\left(\frac{1}{8}\theta_1b-\frac{1}{4}b\right)t_1^6+\frac{1}{5}\left(\frac{1}{8}\theta_1b-\frac{1}{4}b+\frac{1}{3}Ra\theta_1\right)t_1^5\\ +\frac{1}{4}\left(-\frac{1}{3}\theta_2a-\frac{1}{2}Rb\right)t_1^4+\frac{1}{3}\left(\frac{1}{2}b-Ra\right)t_1^3+\frac{1}{2}at_1^2\end{array}\right]$$

$$-cI_p\left[\begin{array}{l}-\frac{1}{6}R\left(\frac{1}{8}\theta_1b-\frac{1}{4}b\right)M^6+\frac{1}{5}\left(\frac{1}{8}\theta_1b-\frac{1}{4}b+\frac{1}{3}Ra\theta_1\right)M^5\\ +\frac{1}{4}\left(-\frac{1}{3}\theta_2a-\frac{1}{2}Rb\right)M^4+\frac{1}{3}\left(\frac{1}{2}b-Ra\right)M^3+\frac{1}{2}aM^2\end{array}\right]$$

$$+cI_p\left[\begin{array}{l}\frac{1}{48}R\theta_2bt_2^6+\frac{1}{5}\left(-\frac{1}{8}\theta_2b+\frac{1}{3}R\theta_2a\right)t_2^5\\ +\frac{1}{4}\left(-\frac{1}{3}\theta_2a-R\left(\frac{1}{2}b+\frac{1}{2}a\theta_2t_1+\frac{1}{4}b\theta_2t_1^2\right)\right)t_2^4\\ +\frac{1}{3}\left(\frac{1}{2}b+\frac{1}{2}a\theta_2t_1+\frac{1}{4}b\theta_2t_1^2-Ra\right)t_2^3\\ +\frac{1}{2}\left(a-R\left(-at_1-\frac{1}{2}bt_1^2-\frac{1}{6}a\theta_2t_1^3-\frac{1}{8}b\theta_2t_1^4\right)\right)t_2^2\\ -\left(at_1+\frac{1}{2}bt_1^2+\frac{1}{6}a\theta_2t_1^3+\frac{1}{8}b\theta_2t_1^4\right)t_2\end{array}\right]$$

$$-cI_p\left[\begin{array}{l}\frac{1}{48}R\theta_2bt_1^6+\frac{1}{5}\left(-\frac{1}{8}\theta_2b+\frac{1}{3}R\theta_2a\right)t_1^5\\ +\frac{1}{4}\left(-\frac{1}{3}\theta_2a-R\left(\frac{1}{2}b+\frac{1}{2}a\theta_2t_1+\frac{1}{4}b\theta_2t_1^2\right)\right)t_1^4\\ +\frac{1}{3}\left(\frac{1}{2}b+\frac{1}{2}a\theta_2t_1+\frac{1}{4}b\theta_2t_1^2-Ra\right)t_1^3\\ +\frac{1}{2}\left(a-R\left(-at_1-\frac{1}{2}bt_1^2-\frac{1}{6}a\theta_2t_1^3-\frac{1}{8}b\theta_2t_1^4\right)\right)t_1^2\\ -\left(at_1+\frac{1}{2}bt_1^2+\frac{1}{6}a\theta_2t_1^3+\frac{1}{8}b\theta_2t_1^4\right)t_1\end{array}\right] \quad (28)$$

Case II : $t_1 \le M \le t_2$:

In this case interest payable is:

$$IP_2=cI_p\left[\begin{array}{l}\int_M^{t_2}I_2(t)e^{-Rt}dt+\int_{t_2}^{t_3}I_3(t)e^{-Rt}dt\\ +\int_M^{t_3}I_4(t)e^{-Rt}dt+\int_{t_3}^{t_4}I_5(t)e^{-Rt}dt\end{array}\right]$$

$$=cI_p\left[\begin{array}{l}-\frac{1}{48}R\theta_2bt_3^6+\frac{1}{5}\left(\frac{1}{8}\theta_2b-\frac{1}{3}R\theta_2a\right)t_3^5\\ +\frac{1}{4}\left(\frac{1}{3}\theta_2a-R\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2t_3-\frac{1}{4}b\theta_2t_3^2\right)\right)t_3^4\\ +\frac{1}{3}\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2t_3-\frac{1}{4}b\theta_2t_3^2+Ra\right)t_3^3\\ +\frac{1}{2}\left(-a-R\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2t_3^3+\frac{1}{8}b\theta_2t_3^4\right)\right)t_3^2\\ +\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2t_3^3+\frac{1}{8}b\theta_2t_3^4\right)t_3\end{array}\right]$$

$$-cI_p\left[\begin{array}{l}-\frac{1}{48}R\theta_2bt_2^6+\frac{1}{5}\left(\frac{1}{8}\theta_2b-\frac{1}{3}R\theta_2a\right)t_2^5\\ +\frac{1}{4}\left(\frac{1}{3}\theta_2a-R\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2t_3-\frac{1}{4}b\theta_2t_3^2\right)\right)t_2^4\\ +\frac{1}{3}\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2t_3-\frac{1}{4}b\theta_2t_3^2+Ra\right)t_2^3\\ +\frac{1}{2}\left(-a-R\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2t_3^3+\frac{1}{8}b\theta_2t_3^4\right)\right)t_2^2\\ +\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2t_3^3+\frac{1}{8}b\theta_2t_3^4\right)t_2\end{array}\right]$$

$$+cI_pW\left[\frac{1}{8}R\theta_1t_3^4-\frac{1}{6}\theta_1t_3^3-\frac{1}{2}R\left(1+\frac{1}{2}\theta_1t_1^2\right)t_3^2+t_3+\frac{1}{2}\theta_1t_1^2t_3\right]$$

$$-cI_pW\left[\frac{1}{8}R\theta_1M^4-\frac{1}{6}\theta_1M^3-\frac{1}{2}R\left(1+\frac{1}{2}\theta_1t_1^2\right)M^2+M+\frac{1}{2}\theta_1t_1^2M\right]$$

$$
+\,cI_p\left[\begin{array}{l}
-\frac{1}{48}R\theta_1 b t_4^6+\frac{1}{5}\left(\frac{1}{8}\theta_1 b-\frac{1}{3}R\theta_1 a\right)t_4^5\\
+\frac{1}{4}\left(\frac{1}{3}\theta_1 a-R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_4^4\\
+\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_4^3\\
+\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_4^2\\
+\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_4
\end{array}\right]
$$

$$
-\,cI_p\left[\begin{array}{l}
-\frac{1}{48}R\theta_1 b t_3^6+\frac{1}{5}\left(\frac{1}{8}\theta_1 b-\frac{1}{3}R\theta_1 a\right)t_3^5\\
+\frac{1}{4}\left(\frac{1}{3}\theta_1 a-R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_3^4\\
+\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_3^3\\
+\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_3^2\\
+\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_3
\end{array}\right]
$$

$$
+\,cI_p\left[\begin{array}{l}
\frac{1}{48}R\theta_2 b t_2^6+\frac{1}{5}\left(-\frac{1}{8}\theta_2 b+\frac{1}{3}R\theta_2 a\right)t_2^5\\
+\frac{1}{4}\left(-\frac{1}{3}\theta_2 a-R\left(\frac{1}{2}b+\frac{1}{2}a\theta_2 t_1+\frac{1}{4}b\theta_2 t_1^2\right)\right)t_2^4\\
+\frac{1}{3}\left(\frac{1}{2}b+\frac{1}{2}a\theta_2 t_1+\frac{1}{4}b\theta_2 t_1^2-Ra\right)t_2^3\\
+\frac{1}{2}\left(a-R\left(-at_1-\frac{1}{2}bt_1^2-\frac{1}{6}a\theta_2 t_1^3-\frac{1}{8}b\theta_2 t_1^4\right)\right)t_2^2\\
-\left(at_1+\frac{1}{2}bt_1^2+\frac{1}{6}a\theta_2 t_1^3+\frac{1}{8}b\theta_2 t_1^4\right)t_2
\end{array}\right]
$$

$$
-\,cI_p\left[\begin{array}{l}
\frac{1}{48}R\theta_2 b M^6+\frac{1}{5}\left(-\frac{1}{8}\theta_2 b+\frac{1}{3}R\theta_2 a\right)M^5\\
+\frac{1}{4}\left(-\frac{1}{3}\theta_2 a-R\left(\frac{1}{2}b+\frac{1}{2}a\theta_2 t_1+\frac{1}{4}b\theta_2 t_1^2\right)\right)M^4\\
+\frac{1}{3}\left(\frac{1}{2}b+\frac{1}{2}a\theta_2 t_1+\frac{1}{4}b\theta_2 t_1^2-Ra\right)M^3\\
+\frac{1}{2}\left(a-R\left(-at_1-\frac{1}{2}bt_1^2-\frac{1}{6}a\theta_2 t_1^3-\frac{1}{8}b\theta_2 t_1^4\right)\right)M^2\\
-\left(at_1+\frac{1}{2}bt_1^2+\frac{1}{6}a\theta_2 t_1^3+\frac{1}{8}b\theta_2 t_1^4\right)M
\end{array}\right] \qquad (29)
$$

Case III : $t_2 \leq M \leq t_3$:

In this case interest payable is:

$$
IP_3 = cI_p\left[\int_M^{t_3} I_3(t)e^{-Rt}dt+\int_M^{t_3} I_4(t)e^{-Rt}dt+\int_{t_3}^{t_4} I_5(t)e^{-Rt}dt\right]
$$

$$
=cI_p\left[\begin{array}{l}
-\frac{1}{48}R\theta_2 b t_3^6+\frac{1}{5}\left(\frac{1}{8}\theta_2 b-\frac{1}{3}R\theta_2 a\right)t_3^5\\
+\frac{1}{4}\left(\frac{1}{3}\theta_2 a-R\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2\right)\right)t_3^4\\
+\frac{1}{3}\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2+Ra\right)t_3^3\\
+\frac{1}{2}\left(-a-R\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2 t_3^3+\frac{1}{8}b\theta_2 t_3^4\right)\right)t_3^2\\
+\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2 t_3^3+\frac{1}{8}b\theta_2 t_3^4\right)t_3
\end{array}\right]
$$

$$
-\,cI_p\left[\begin{array}{l}
-\frac{1}{48}R\theta_2 b M^6+\frac{1}{5}\left(\frac{1}{8}\theta_2 b-\frac{1}{3}R\theta_2 a\right)M^5\\
+\frac{1}{4}\left(\frac{1}{3}\theta_2 a-R\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2\right)\right)M^4\\
+\frac{1}{3}\left(-\frac{1}{2}b-\frac{1}{2}a\theta_2 t_3-\frac{1}{4}b\theta_2 t_3^2+Ra\right)M^3\\
+\frac{1}{2}\left(-a-R\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2 t_3^3+\frac{1}{8}b\theta_2 t_3^4\right)\right)M^2\\
+\left(at_3+\frac{1}{2}bt_3^2+\frac{1}{6}a\theta_2 t_3^3+\frac{1}{8}b\theta_2 t_3^4\right)M
\end{array}\right]
$$

$$
+cI_pW\left[\frac{1}{8}R\theta_1 t_3^4-\frac{1}{6}\theta_1 t_3^3-\frac{1}{2}R\left(1+\frac{1}{2}\theta_1 t_1^2\right)t_3^2+t_3+\frac{1}{2}\theta_1 t_1^2 t_3\right]
$$

$$
-cI_pW\left[\frac{1}{8}R\theta_1 M^4-\frac{1}{6}\theta_1 M^3-\frac{1}{2}R\left(1+\frac{1}{2}\theta_1 t_1^2\right)M^2+M+\frac{1}{2}\theta_1 t_1^2 M\right]
$$

$$
+\,cI_p\left[\begin{array}{l}
-\frac{1}{48}R\theta_1 b t_4^6+\frac{1}{5}\left(\frac{1}{8}\theta_1 b-\frac{1}{3}R\theta_1 a\right)t_4^5\\
+\frac{1}{4}\left(\frac{1}{3}\theta_1 a-R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_4^4\\
+\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_4^3\\
+\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_4^2\\
+\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_4
\end{array}\right]
$$

$$- cI_p \begin{bmatrix} -\frac{1}{48}R\theta_1 bt_3^6+\frac{1}{5}\left(\frac{1}{8}\theta_1 b-\frac{1}{3}R\theta_1 a\right)t_3^5 \\ +\frac{1}{4}\left(\frac{1}{3}\theta_1 a -R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bT^2+aT\right)-\frac{1}{2}b\right)\right)t_3^4 \\ +\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bT^2+aT\right)-\frac{1}{2}b+Ra\right)t_3^3 \\ +\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_3^2 \\ +\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_3 \end{bmatrix} \quad (30)$$

Case IV : $t_3 \le M \le t_4$:

In this case interest payable is:

$$IP_4 = cI_p\left[\int_{t_3}^{t_4} I_5(t)e^{-Rt}dt\right]$$

$$= cI_p \begin{bmatrix} -\frac{1}{48}R\theta_1 bt_4^6+\frac{1}{5}\left(\frac{1}{8}\theta_1 b-\frac{1}{3}R\theta_1 a\right)t_4^5 \\ +\frac{1}{4}\left(\frac{1}{3}\theta_1 a -R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)t_4^4 \\ +\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)t_4^3 \\ +\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)t_4^2 \\ +\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)t_4 \end{bmatrix}$$

$$- cI_p \begin{bmatrix} -\frac{1}{48}R\theta_1 bM^6+\frac{1}{5}\left(\frac{1}{8}\theta_1 b-\frac{1}{3}R\theta_1 a\right)M^5 \\ +\frac{1}{4}\left(\frac{1}{3}\theta_1 a -R\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b\right)\right)M^4 \\ +\frac{1}{3}\left(-\frac{1}{2}\theta_1\left(\frac{1}{2}bt_4^2+at_4\right)-\frac{1}{2}b+Ra\right)M^3 \\ +\frac{1}{2}\left(-a-R\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)\right)M^2 \\ +\left(\frac{1}{8}b\theta_1 t_4^4+\frac{1}{6}a\theta_1 t_4^3+\frac{1}{2}bt_4^2+at_4\right)M \end{bmatrix} \quad (31)$$

Case V: $M > t_4$:

In this case, no interest charges are paid for the item. So,

$IP_5 = 0$. (32)

The retailer's total cost during a cycle, $TC_i(t_2,t_3,t_5)$, i=1,2,3,4,5 consisted of the following:

$$TC_i=\frac{1}{T}\left[A + HC(OW) + HC(RW) + DC + SC + IP_i - IE_i\right] \quad (33)$$

and t_1, t_4 and T are approximately related to t_2, t_3 and t_5 through equations (16), (18) and (20) respectively.

Substituting values from equations (21) to (25) and equations (26) to (32) in equation (33), total costs for the five cases will be as under:

$$TC_1=\frac{1}{T}\left[A + HC(OW) + HC(RW) + DC + SC + IP_1 - IE_1\right] \quad (34)$$

$$TC_2=\frac{1}{T}\left[A + HC(OW) + HC(RW) + DC + SC + IP_2 - IE_1\right] \quad (35)$$

$$TC_3=\frac{1}{T}\left[A + HC(OW) + HC(RW) + DC + SC + IP_3 - IE_1\right] \quad (36)$$

$$TC_4=\frac{1}{T}\left[A + HC(OW) + HC(RW) + DC + SC + IP_4 - IE_1\right] \quad (37)$$

$$TC_5=\frac{1}{T}\left[A + HC(OW) + HC(RW) + DC + SC + IP_5 - IE_2\right] \quad (38)$$

The optimal value of $t_2= t_2^*$, $t_3= t_3^*$ and $t_5= t_5^*$ (say), which minimizes TC_it_2, t_3,t_5) can be obtained by solving equation (34), (35), (36), (37) and (38) by differentiating it with respect to t_2, t_3 and t_5 and equate it to zero

i.e. $$\frac{\partial TC_i(t_2,t_3,t_5)}{dt_2}=0,\ \frac{\partial TC_i(t_2,t_3,t_5)}{dt_3}=0,\ \frac{\partial TC_i(t_2,t_3,t_5)}{dt_5}=0, \quad (39)$$

provided it satisfies the condition

$$\begin{vmatrix} \frac{\partial T^2C_i(t_2,t_3,t_5)}{\partial t_2^2} & \frac{\partial T^2C_i(t_2,t_3,t_5)}{\partial t_2\partial t_3} & \frac{\partial TC_i(t_2,t_3,t_5)}{\partial t_2\partial t_5} \\ \frac{\partial T^2C_i(t_2,t_3,t_5)}{\partial t_3\partial t_2} & \frac{\partial T^2C_i(t_2,t_3,t_5)}{\partial t_3^2} & \frac{\partial TC_i(t_2,t_3,t_5)}{\partial t_3\partial t_5} \\ \frac{\partial TC_i(t_2,t_3,t_5)}{\partial t_5\partial t_2} & \frac{\partial TC_i(t_2,t_3,t_5)}{\partial t_5\partial t_3} & \frac{\partial T^2C_i(t_2,t_3,t_5)}{\partial t_5^2} \end{vmatrix} > 0. \quad (40)$$

4. NUMERICAL EXAMPLES:

Case I: Considering A= Rs.150, W = 40, k=2, a = 200, b=0.05, c=Rs. 10, p= Rs. 15, c2 = Rs. 8, θ_1=0.1, θ_2 =0.06, x_1 = Rs. 1, y_1=0.05, x_2= Rs. 3, y_2=0.06, Ip= Rs. 0.15, Ie= Rs. 0.12, R = 0.06, M=0.30 year, in appropriate units. The optimal value of t_2^*=0.5251, t_3^*=0.6425, t_5^*=0.9908 and TC_1^*= Rs. 221.4127.

Case II: Considering A= Rs.150, W = 50, k=2, a = 200, b=0.05, c=Rs. 10, p= Rs. 15, c2 = Rs. 8, θ_1=0.1, θ_2 =0.06, x_1 = Rs. 1, y_1=0.05, x_2= Rs. 3, y_2=0.06, Ip= Rs. 0.15, Ie= Rs. 0.12, R = 0.06, M=0.468 year, in appropriate units. The optimal value of t_2^*=0.483 8, t_3^*=0.5591, t_5^*=0.8818 and TC_2^*= Rs.185.3407.

Case III: Considering A= Rs.150, W = 50, k=2, a = 200, b=0.05, c=Rs. 10, p= Rs. 15, c2 = Rs. 8, θ_1=0.1, θ_2 =0.06, x_1 = Rs. 1, y_1=0.05, x_2= Rs. 3, y_2=0.06, Ip= Rs. 0.15, Ie= Rs. 0.12, R = 0.06, M=0.53 year, in appropriate units. The optimal value of t_2^*=0.4741, t_3^*=0.5512, t_5^*=0.8638 and TC_3^*= Rs. 169.7496.

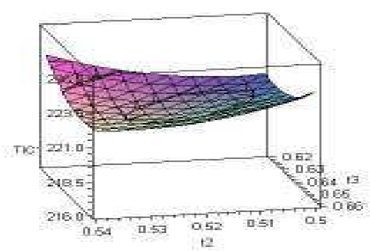
Case IV: Considering A= Rs.150, W = 50, k=2, a = 200, b=0.05, c=Rs. 10, p= Rs. 15, c2 = Rs. 8, θ_1=0.1, θ_2 =0.06, x_1 = Rs. 1, y_1=0.05, x_2= Rs. 3, y_2=0.06, Ip= Rs. 0.15, Ie= Rs. 0.12, R = 0.06, M=0.70 year, in appropriate units. The optimal value of t_2^*=0.4811, t_3^*=0.5602, t_5^*=0.8425 and TC_4^*= Rs. 124.3952.

Case V: Considering A= Rs.150, W = 50, k=2, a = 200, b=0.05, c=Rs. 10, p= Rs. 15, c2 = Rs. 8, θ_1=0.1, θ_2 =0.06, x_1 = Rs. 1, y_1=0.05, x_2= Rs. 3, y_2=0.06, Ip= Rs. 0.15, Ie= Rs. 0.12, R = 0.06, M=0.80 year, in appropriate units. The optimal value of t_2^*=0.4601, t_3^*=0.5333, t_5^*=0.7954 and TC_5^*= Rs. 94.6328.

The second order conditions given in equation (40) are also satisfied. The graphical representation of the convexity of the cost functions for the five cases are also given.

Case I

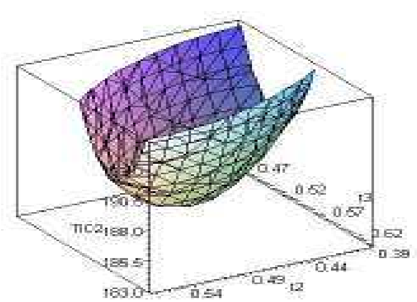
t_2, t_3 and cost



Graph 1

Case II

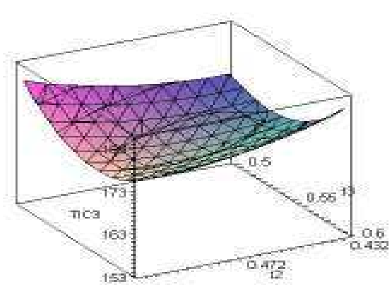
t_2, t_3 and cost



Graph 2

Case III

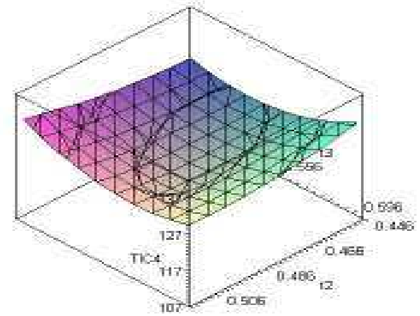
t_2, t_3 and cost



Graph 3

Case IV

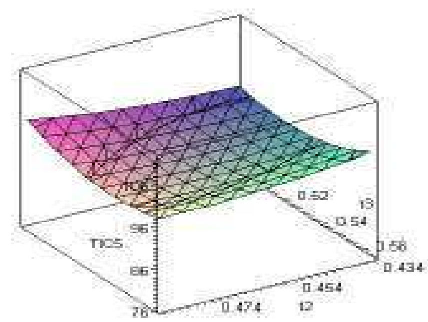
t_2, t_3 and cost



Graph 4

Case V

t_2, t_3 and cost



Graph 5

5. SENSITIVITY ANALYSIS:

On the basis of the data given in example above we have studied the sensitivity analysis by changing the following parameters one at a time and keeping the rest fixed.

Table 1
Sensitivity Analysis
Case I ($0 \leq M \leq t_1$)

Para-meter	%	t_2	t_3	t_5	Cost
x_1	+10%	0.5178	0.6341	0.9856	223.8276
	+5%	0.5215	0.6388	0.9881	222.6624
	-5%	0.5280	0.6460	0.9934	220.1537
	-10%	0.5323	0.6501	0.9970	218.9882
x_2	+10%	0.5230	0.6363	0.9855	222.0427
	+5%	0.5241	0.6394	0.9880	221.7353
	-5%	0.5260	0.6456	0.9930	221.0730
	-10%	0.5267	0.6490	0.9961	220.7341
θ_1	+10%	0.5195	0.6356	0.9850	222.3297
	+5%	0.5225	0.6390	0.9880	222.0821
	-5%	0.5276	0.6460	0.9938	220.9426
	-10%	0.5301	0.6495	0.9972	220.4681
θ_2	+10%	0.5247	0.6418	0.9907	221.4800
	+5%	0.5249	0.6421	0.9907	221.4462
	-5%	0.5253	0.6429	0.9909	221.3792
	-10%	0.5255	0.6433	0.9910	221.3458
R	+10%	0.5271	0.6451	0.9938	221.0637
	+5%	0.5260	0.6436	0.9923	221.2369
	-5%	0.5241	0.6411	0.9892	221.5858
	-10%	0.5230	0.6400	0.9875	221.7564

A	+10%	0.5172	0.6348	0.9901	234.4636
	+5%	0.5212	0.6382	0.9904	227.9369
	-5%	0.5292	0.6469	0.9911	214.8999
	-10%	0.5322	0.6520	0.9915	208.3834
M	+10%	0.5415	0.6640	1.0090	216.4837
	+5%	0.5349	0.6535	0.9999	219.0070
	-5%	0.5143	0.6312	0.9817	223.7747
	-10%	0.5035	0.6199	0.9719	226.1033

Table 2
Sensitivity Analysis
Case II ($t_1 \le M \le t_2$)

Para-meter	%	t_2	t_3	t_5	Cost
x_1	+10%	0.5876	0.6669	0.9938	189.8242
	+5%	0.5507	0.6280	0.9528	187.6192
	-5%	0.4275	0.5070	0.8296	183.6355
	-10%	0.2912	0.3749	0.6950	181.6595
x_2	+10%	0.4152	0.4868	0.8086	185.0084
	+5%	0.4495	0.5229	0.8452	185.1574
	-5%	0.5375	0.6148	0.9383	185.8100
	-10%	0.5525	0.6315	0.9550	185.8414
θ_1	+10%	0.5637	0.6448	0.9714	195.5348
	+5%	0.5244	0.6019	0.9264	186.3545
	-5%	0.4400	0.5164	0.8386	184.4813
	-10%	0.3961	0.4736	0.7950	183.7685
θ_2	+10%	0.4834	0.5581	0.8410	185.4209
	+5%	0.4836	0.5586	0.8814	185.4074
	-5%	0.4840	0.5595	0.8824	185.3801
	-10%	0.4842	0.5599	0.8827	185.3663
R	+10%	0.4666	0.5420	0.8651	184.9778
	+5%	0.4752	0.5504	0.8733	185.1554
	-5%	0.4924	0.5677	0.8905	185.5344
	-10%	0.5010	0.5762	0.8988	185.7364
A	+10%	0.0963	0.2071	0.5431	209.5495
	+5%	0.2133	0.3050	0.6327	195.6693
	-5%	0.6544	0.7293	1.0495	179.3218
	-10%	0.7506	0.8255	1.1432	174.2513
M	+10%	0.7578	0.8817	1.2150	184.8290
	+5%	0.6792	0.7765	1.1071	184.8709
	-5%	0.2539	0.3432	0.6675	191.9256
	-10%	0.0238	0.1326	0.4650	205.0787

Table 3
Sensitivity Analysis
Case III ($t_2 \le M \le t_3$)

Para-meter	%	t_2	t_3	t_5	Cost
x_1	+10%	0.4721	0.5524	0.8664	172.3117
	+5%	0.4731	0.5518	0.8657	169.7473
	-5%	0.4752	0.5508	0.8620	168.4594
	-10%	0.4763	0.5504	0.8603	167.1628
x_2	+10%	0.4747	0.5505	0.8647	170.1040
	+5%	0.4744	0.5508	0.8642	169.9283
	-5%	0.4739	0.5516	0.8634	169.5684
	-10%	0.4736	0.5521	0.8631	169.3816
θ_1	+10%	0.4721	0.5501	0.8632	170.6716
	+5%	0.4731	0.5506	0.8435	170.2118
	-5%	0.4751	0.5519	0.8641	169.2850
	-10%	0.4760	0.5524	0.8644	168.8168
θ_2	+10%	0.4738	0.5508	0.8636	169.7798
	+5%	0.4740	0.5510	0.8637	169.7652
	-5%	0.4742	0.5514	0.8639	169.7341
	-10%	0.4743	0.5516	0.8640	169.7184
R	+10%	0.4775	0.5546	0.8684	169.6702
	+5%	0.4758	0.5529	0.8663	169.7101
	-5%	0.4724	0.5493	0.8612	169.7898
	-10%	0.4707	0.5470	0.8587	169.8299

A	+10%	0.5141	0.6003	0.9242	185.2697
	+5%	0.4961	0.5794	0.8983	177.6358
	-5%	0.4521	0.5231	0.8279	159.0316
	-10%	0.4336	0.5020	0.8030	153.4762
M	+10%	0.4703	0.5462	0.8490	156.0239
	+5%	0.4722	0.5487	0.8564	162.9301
	-5%	0.4760	0.5535	0.8710	176.4851
	-10%	0.4778	0.5567	0.8792	183.1441

Table 4
Sensitivity Analysis
Case IV ($t_3 \le M \le T$)

Para-meter	%	t_2	t_3	t_5	Cost
x_1	+10%	0.4743	0.5556	0.8400	127.0737
	+5%	0.4778	0.5578	0.8412	125.7420
	-5%	0.4843	0.5626	0.8438	123.0332
	-10%	0.4879	0.5648	0.8450	121.6602
x_2	+10%	0.4848	0.5592	0.8418	124.8095
	+5%	0.4825	0.5594	0.8422	124.6008
	-5%	0.4797	0.5608	0.8429	124.1790
	-10%	0.4783	0.5615	0.8432	123.9520
θ_1	+10%	0.4807	0.5601	0.8430	125.4209
	+5%	0.4809	0.5601	0.8428	124.9084
	-5%	0.4813	0.5602	0.8422	123.8806
	-10%	0.4815	0.5603	0.8420	123.3654
θ_2	+10%	0.4854	0.5653	0.8484	124.5065
	+5%	0.4833	0.5628	0.8454	124.4491
	-5%	0.4790	0.5577	0.8398	124.3464
	-10%	0.4769	0.5551	0.8370	124.3014
R	+10%	0.4747	0.5520	0.8343	124.4188
	+5%	0.4779	0.5561	0.8381	124.4014
	-5%	0.4816	0.5607	0.8430	124.3533
	-10%	0.4822	0.5612	0.8435	124.3132
A	+10%	0.5149	0.6000	0.8922	140.6670
	+5%	0.4943	0.5776	0.8652	132.5913
	-5%	0.4679	0.5428	0.8200	115.9464
	-10%	0.4473	0.5167	0.7874	107.1111
M	+10%	0.4769	0.5550	0.8239	104.3433
	+5%	0.4790	0.5576	0.8332	114.4683
	-5%	0.4831	0.5628	0.8520	134.1335
	-10%	0.4850	0.5654	0.8617	143.6945

Table 5
Sensitivity Analysis
Case V ($M \ge T$)

Para-meter	%	t_2	t_3	t_5	Cost
x_1	+10%	0.4502	0.5245	0.7882	97.3471
	+5%	0.4553	0.5291	0.7922	95.9960
	-5%	0.4647	0.5374	0.7987	93.2562
	-10%	0.4700	0.5422	0.8030	91.8756
x_2	+10%	0.4589	0.5270	0.7893	94.9762
	+5%	0.4595	0.5318	0.7925	94.8100
	-5%	0.4606	0.5364	0.7986	94.4485
	-10%	0.4613	0.5396	0.8019	94.2571
θ_1	+10%	0.4539	0.5256	0.7875	95.5882
	+5%	0.4569	0.5294	0.7914	95.1093
	-5%	0.4633	0.5373	0.7995	94.1566
	-10%	0.4665	0.5412	0.8034	93.6803
θ_2	+10%	0.4577	0.5304	0.7921	94.6474
	+5%	0.4589	0.5319	0.7938	94.6394
	-5%	0.4612	0.5348	0.7971	94.6264
	-10%	0.4624	0.5362	0.7987	94.6223
R	+10%	0.4583	0.5310	0.7933	94.8114
	+5%	0.4594	0.5321	0.7943	94.7242
	-5%	0.4608	0.5344	0.7964	94.5416
	-10%	0.4616	0.5356	0.7974	94.4515

A	+10%	0.4861	0.5666	0.8404	111.8344
	+5%	0.4721	0.5485	0.8166	103.3500
	-5%	0.4481	0.5181	0.7742	85.6373
	-10%	0.4341	0.5001	0.7503	76.3167
M	+10%	0.4660	0.5403	0.7861	69.5642
	+5%	0.4638	0.5381	0.7922	82.2315
	-5%	0.4564	0.5286	0.7986	106.7945
	-10%	0.4542	0.5263	0.8046	118.7377

From the table we observe that as parameters x_1, θ_1 and A increase/ decrease, average total cost increase/ decrease for case I, case II, case III, case IV and case V respectively.

From the table we observe that with increase/ decrease in parameter x_2, there is increase/ decrease in total cost for case I, case III, case IV and case V respectively, whereas for case II with increase/crease in parameter x_2, there is corresponding decrease/ increase in total cost for case II.

Also, we observe that with increase and decrease in the value of θ_2, there is corresponding almost no change in total cost for case I, case II, case III, case IV and case V respectively.

From the table we observe that as parameters R and M increase/ decrease, average total cost decrease/ increase for case I, case II, case III, case IV and case V respectively.

6. CONCLUSION:

In this paper, we have developed a two warehouse inventory model for deteriorating items with linear demand, shortages under inflation and permissible delay in payments. It is assume that rented warehouse holding cost is greater than own warehouse holding cost but provides a better storage facility and there by deterioration rate is low in rented warehouse. Sensitivity with respect to parameters have been carried out. The results show that with the increase/ decrease in the parameter values there is corresponding increase/ decrease in the value of cost.

REFERENCES

[1] Aggarwal, S.P. and Jaggi, C.K. (1995): Ordering policies for deteriorating items under permissible delay in payments; J. Oper. Res. Soc., Vol. 46, pp. 658-662.

[2] Benkherouf, L. (1997): A deterministic order level inventory model for deteriorating items with two storage facilities; International J. Production Economics; Vol. 48, pp. 167-175.

[3] Bhunia, A.K. and Maiti, M. (1998): A two-warehouse inventory model for deteriorating items with a linear trend in demand and shortages; J. of Oper. Res. Soc.; Vol. 49, pp. 287-292.

[4] Bhunia, A.K., Shaikh, A.A. and Gupta, R.K. (2013): A study on two-warehouse partial backlogged deteriorating inventory models under inflation via particle swarm optimization; International J. of System Sciences, pp. 1-15.

[5] Chang, C.T., Teng, J.T. and Goyal, S.K. (2008): Inventory lot sizing models under trade credits; Asia Pacific J. Oper. Res., Vol. 25, pp. 89-112.

[6] Chung, K.J. and Huang, T.S. (2007): The optimal retailer's ordering policies for deteriorating items with limited storage capacity under trade credit financing; International J. Production Economics; Vol. 106, pp. 127-145.

[7] Covert, R.P. and Philip, G.C. (1973): An EOQ model for items with Weibull distribution deterioration; American Institute of Industrial Engineering Transactions, Vol. 5, pp. 323-328.

[8] Ghare, P.M. and Schrader, G.F. (1963): A model for exponentially decaying inventories; J. Indus. Engg., Vol. 14, pp. 238-243.

[9] Goyal, S.K. (1985): Economic order quantity under conditions of permissible delay in payments, J. O.R. Soc., Vol. 36, pp. 335-338.

[10] Goyal, S.K. and Giri, B.C. (2001): Recent trends in modeling of deteriorating inventory; Euro. J. O.R., Vol. 134, pp. 1-16.

[11] Hartley, R.V. (1976): Operations research – a managerial emphasis; Good Year, Santa Monica, CA, Chapter 12, pp. 315-317.

[12] Jamal, A.M.M., Sarker, B.R. and Wang, S. (1997): An ordering policy for deteriorating items with allowable shortages and permissible delay in payment; J. Oper. Res. Soc., Vol. 48, pp. 826-833.

[13] Kar, S., Bhunia, A.K. and Maiti, M. (2001): Deterministic inventory model with two levels of storage, a linear trend in demand and a fixed time horizon; Computers and Oper. Res.; Vol. 28, pp. 1315-1331.

[14] Liang and Zhou (2011): A two warehouse inventory model for deteriorating items under conditionally permissible delay in payments, Applied Mathematical Modeling, Vol. 35, pp. 2221-2231.

[15] Min, J., Zhou, Y.W., Liu, G.Q. and Wang, S.D. (2012): An EPQ model for deteriorating items with inventory level dependent demand and permissible delay in payments; International J. of System Sciences, Vol. 43, pp. 1039-1053.

[16] Nahmias, S. (1982): Perishable inventory theory: a review; Operations Research, Vol. 30, pp. 680-708.

[17] Ouyang, L. Y., Wu, K.S. and Yang, C.T. (2006): A study on an inventory model for non-instantaneous deteriorating items with permissible delay in payments; Computers and Industrial Engineering, Vol. 51, pp. 637-651.

[18] Patel, R. and Parekh, R.U. (2014): Inventory model for variable deteriorating items with two warehouses under shortages, time varying holding cost, inflation and permissible delay in payments; International Referred J. of Engg. And Science, Vol. 3(8), pp. 6-17.

[19] Raafat, F. (1991): Survey of literature on continuously deteriorating inventory model, Euro. J. of O.R. Soc., Vol. 42, pp. 27-37.

[20] Sarma, K.V.S. (1987): A deterministic inventory model for deteriorating items with two storage facilities; Euro. J. O.R., Vol. 29, pp. 70-72.

[21] Shah, Y.K. (1997): An order level lot size inventory for deteriorating items; American Institute of Industrial Engineering Transactions, Vol. 9, pp. 108-112.

[22] Teng, J.T., Chang, C.T. and Goyal, S.K. (2005): Optimal pricing and ordering policy under permissible delay in payments; International J. of Production Economics, Vol. 97, pp. 121-129.

[23] Wu, K.S., Ouyang, L. Y. and Yang, C.T. (2006): An optimal replenishment policy for non-instantaneous deteriorating items with stock dependent demand and partial backlogging; International J. of Production Economics, Vol. 101, pp. 369-384.

[24] Yang, H.L. (2004): Two warehouse inventory models for deteriorating items with shortages under inflation; Euro. J. Oper. Res.; Vol. 157, pp. 344-356.