φ-Confomally Flat Generalized Sasakian Space Form

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Abstract: - **In this article we studied generalized Sasakian space forms which are ϕ–conformally flat, ϕ–conharmonically flat, ϕ–projectively flat and prove that they are η-Einstein manifolds under suitable assumptions.**

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I. INTRODUCTION

In [**3**] authors studied ϕ–conformally flat contact metric manifolds under the condition that the characteristic vector field ξ belongs to (k,μ)-nullity distribution.C. Özgür [**7**] studied ϕ–conformally flat Lorentzian para-Sasakian manifolds. In [**8**] U. K. Kim studied generalized Sasakian space forms and proved a classification theorem under the assumption that the characteristic vector field is killing. In this chapter, we shall study ϕ–conformally flat generalized Sasakian space forms with $Q\phi=\phi Q$, Q being the Ricci operator of the manifold

II. PRELIMINARIES

A $(2n+1)$–dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field ϕ, a vector field ξ and a 1– form η such that

(2.1) $$\eta(\xi) = 1, \ \phi^2 X = -X + \eta(X)\xi \quad and$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\,\eta(Y),$$

for any vector fields X, Y on M. Then, $\phi(\xi) = 0$ and $\eta o \phi = 0$. Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$ where Φ is defined as $\Phi(X,Y) = g(X,\phi Y)$ is fundamental 2–form of M.

An almost contact metric manifold is called a Sasakian manifold if

(2.2) $$(\nabla_X \phi)\, Y = g(X, Y)\,\xi - \eta(Y)\, X, \ \nabla_X \xi = -\phi X,$$

for any X,Y on TM, where ∇ denotes the Riemannian connection of g.

In [**1**] Alegre, Blair and Carriazo introduced the notion of a generalized Sasakian space form. Given an almost contact metric manifold $M(\phi,\xi,\eta,g)$, we say that M is a generalized Sasakian space form denoted by $M(f_1, f_2, f_3)$ if there exist three functions f_1, f_2, *and* f_3 on M such that,

(2.3) $$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{ g(X, \phi Z)\,\phi Y - g(Y, \phi Z)\,\phi X + 2g(X, \phi Y)\,\phi Z\} \\ &+ f_3\{\eta(X)\,\eta(Z)\,Y - \eta(Y)\eta(Z)X + g(X, Z)\,\eta(Y)\,\xi \\ &- g(Y,Z)\,\eta(X)\,\xi\}, \end{aligned}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M .This kind of manifold appears as a natural generalization of the well–known Sasakian space form, which can be obtained as a particular case of generalized Sasakian space forms by taking $f_1 = \dfrac{c+3}{4}$, $f_2 = f_3 = \dfrac{c-1}{4}$. On a generalized Sasakian space form we have

$$QX = (2nf_1 + 3f_2 - f_3)X \\ -(3f_2 + (2n-1)f_3)\eta(X)\xi \text{'}$$

(2.5) $$\tau = 2n(2n+1)f_1 + 6nf_2 - 4nf_3,$$

(2.6) $R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}$.From (2.6), we have

(2.7) $$R(X,\xi)\xi = (f_1 - f_3)\{X - \eta(X)\xi\}.$$

Using (2.4) and (2.6) we have

(2.8) $$S(X,\xi) = 2n(f_1 - f_3)\eta(X),$$

(2.9) $$S(X,Y) = (2nf_1 + 3f_2 - f_3)\,g(X,Y) \\ -(3f_2 + (2n-1)f_3)\eta(X)\eta(Y)$$

(2.10) $$S(\phi X,\phi Y) = S(X,Y) \\ -2n(f_1 - f_3)\eta(X)\eta(Y).$$

A generalized Sasakian space form $M(f_1, f_2, f_3)$ is said to be η-Einstein if its Ricci tensor S is of the form

(2.11) $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$,

for any vector fields X and Y, where a, b are smooth functions on M. Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian space form. The Weyl conformal curvature tensor C, the conharmonic curvature tensor K and the projective curvature tensor P of $M(f_1, f_2, f_3)$ are defined by

(2.12)
$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{\tau}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y],$$

(2.13)
$$K(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY],$$

(2.14)
$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[g(Y,Z)QX - g(X,Z)QY],$$

respectively, where Q is the Ricci operator, defined by $S(X,Z) = g(QX,Y)$, S is the Ricci tensor, $\tau = tr(S)$ is the scalar curvature and $X,Y,Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M. Let C be the Weyl conformal curvature tensor of M. Since at each point $p \in M$ the tangent space $T_p(M)$ can be decomposed into direct sum $T_p(M) = \phi(T_p(M)) + L(\xi_p)$,

where $L(\xi_p)$ is a 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p, we have map:
$$C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) + L(\xi_p).$$

It is natural to consider the following particular cases:

(1) $C : T_p(M) \times T_p(M) \times T_p(M) \to L(\xi_p)$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero.

(2) $C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M))$, that is, the projection of the image of C in $L(\xi_p)$ is zero.

(3) $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \to L(\xi_p)$, that is, when C is restricted to $\phi(T_p(M))$ is zero. This condition is equivalent to

(2.15) $\phi^2 C(\phi X, \phi Y)\phi Z = 0$, see ([**3**], [**5**]).

The case (1) and (2) were considered in [**9**] and [**10**] respectively. The case (3) was considered in [**3**], [**5**]. Now our aim is to study generalized Sasakian space forms satisfying (2.15).

III. MAIN RESULTS

In this section we consider ϕ–conformally flat, ϕ–conharmonically flat, ϕ–projectively flat generalized Sasakian space forms.

Definition1 ([2]) *A differentiable manifold (M,g) satisfying the condition (2.15) is called ϕ–conformally flat.*

Theorem 1 ***Let $M(f_1, f_2, f_3)$ be a (2n+1)-dimensional generalized Sasakian space form, which is ϕ-conformally flat. If M is contact metric manifold with $\phi Q = Q\phi$, then it is η-Einstein manifold.***

Proof: Suppose M be a (2n+1)-dimensional, ϕ-conformally flat generalized Sasakian space form then, it is easy to see that $\phi^2 C(\phi X, \phi Y)\phi Z = 0$ holds if and only if

(3.1) $g(C(\phi X, \phi Y)\phi Z, \phi W) = 0$, for any vector fields $X,Y,Z,W \in \chi(M)$. Using (2.12) ϕ-conformally flat means

(3.2)
$$g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n-1}[g(Q\phi Y, \phi Z)g(\phi X, \phi W) + g(\phi Y, \phi Z)g(Q\phi X, \phi W) - g(Q\phi X, \phi Z)g(\phi Y, \phi W) - g(\phi X, \phi Z)g(Q\phi Y, \phi W)] - \frac{\tau}{2n(2n-1)}\begin{bmatrix} g(\phi Y, \phi Z)g(\phi X, \phi W) \\ - g(\phi X, \phi Z)g(\phi Y, \phi W) \end{bmatrix}.$$

Let $\{e_1,....,e_{2n},\xi\}$ be a local orthonormal basis of vector fields in M. By using that $\{\phi e_1,....,\phi e_{2n},\xi\}$ is also a local orthonormal basis, if we put $X = W = e_\alpha$ in (3.2) and sum up with respect to α, then

(3.3)

$$\sum_{\alpha=1}^{2n} g\big(R(\phi e_\alpha,\phi Y)\phi Z,\phi e_\alpha\big)=\frac{1}{2n-1}\sum_{\alpha=1}^{2n}\big[g(Q\phi Y,\phi Z)g(\phi e_\alpha,\phi e_\alpha)$$
$$+g(\phi Y,\phi Z)g(Q\phi e_\alpha,\phi e_\alpha)-g(Q\phi e_\alpha,\phi Z)g(\phi Y,\phi e_\alpha)-g(\phi e_\alpha,\phi Z)g(Q\phi Y,\phi e_\alpha)\big]$$
$$-\frac{\tau}{2n(2n-1)}\sum_{\alpha=1}^{2n}\begin{bmatrix} g(\phi Y,\phi Z)g(\phi e_\alpha,\phi e_\alpha) \\ -g(\phi e_\alpha,\phi Z)g(\phi Y,\phi e_\alpha)\end{bmatrix}.$$

It is easy to verify that

(3.4)

$$\sum_{\alpha=1}^{2n} g\big(R(\phi e_\alpha,\phi Y)\phi Z,\phi e_\alpha\big)=S\big(\phi Y,\phi Z\big)-(f_1-f_3)\,g(\phi Y,\phi Z)$$

,

(3.5) $$\sum_{\alpha=1}^{2n} S\big(\phi e_\alpha,\phi e_\alpha\big)=\tau-2n(f_1-f_3),$$

(3.6) $$\sum_{\alpha=1}^{2n} g\big(\phi e_\alpha,\phi Z\big)S\big(\phi Y,\phi e_\alpha\big)=S\big(\phi Y,\phi Z\big),$$

(3.7) $$\sum_{\alpha=1}^{2n} g\big(\phi e_\alpha,\phi e_\alpha\big)=2n,$$

And

(3.8) $$\sum_{\alpha=1}^{2n} g\big(\phi e_\alpha,\phi Z\big)g\big(\phi Y,\phi e_\alpha\big)=g\big(\phi Y,\phi Z\big).$$

In view of (3.4) – (3.8) the equation (3.3) can be written as

(3.9) $$S\big(\phi Y,\phi Z\big)=\left(\frac{\tau}{2n}-(f_1-f_3)\right)g\big(\phi Y,\phi Z\big).$$

Now, by using (2.1) and (2.10), the equation (2.24) takes the form

(3.10)
$$S\big(Y,Z\big)=\left(\frac{\tau}{2n}-(f_1-f_3)\right)g\big(Y,Z\big)-$$
$$\left(\frac{\tau}{2n}-(2n+1)(f_1-f_3)\right)\eta(Y)\eta(Z)$$

which implies, M is an η-Einstein manifold. This completes the proof of the theorem.Using (2.9) and (3.10) we have

Corollary 1 *Let* $M(f_1,f_2,f_3)$ *be a (2n+1)-dimensional generalized Sasakian space form. If* M *is contact metric manifold with* $\phi Q = Q\phi$, then f_1, f_2 *and* f_3 are connected by the relation $\frac{\tau}{2n}=(2n+1)f_1+3f_2-2f_3$.

Definition 2([4]) *A differentiable manifold (*M*, g), satisfying the condition*

(3.11) $$\phi^2K\big(\phi X,\phi Y\big)\phi Z=0$$ *is called ϕ-conharmonically flat*. In [**2**] authors considered (k-μ)-contact manifolds satisfying (3.11). Now, we will study the condition (2.26) on a generalized Sasakian space form.

Theorem 2 *Let* $M(f_1,f_2,f_3)$ *be a (2n+1)-dimensional generalized Sasakian space form, which is ϕ-conharmonically flat. If* M *is contact metric manifold with $\phi Q = Q\phi$, then it is* η*-Einstein manifold with zero scalar curvature.*

Proof: Suppose M be a (2n+1)-dimensional, ϕ-conharmonically flat generalized Sasakian space form then, it is easily seen that $\phi^2K\big(\phi X,\phi Y\big)\phi Z=0$ holds if and only if

(3.12) $g\big(K(\phi X,\phi Y)\phi Z,\phi W\big)=0$, for any vector fields $X,Y,Z,W\in\chi(M)$.Using (2.13) ϕ-conharmonically flat means

(3.13)
$$g\big(R(\phi X,\phi Y)\phi Z,\phi W\big)$$
$$=\frac{1}{2n-1}\big[g(Q\phi Y,\phi Z)g(\phi X,\phi W)$$
$$+g(\phi Y,\phi Z)g(Q\phi X,\phi W)$$
$$-g(Q\phi X,\phi Z)g(\phi Y,\phi W)-g(\phi X,\phi Z)g(Q\phi Y,\phi W)\big]$$

Similar to the proof of Theorem 1, we can suppose that $\{e_1,....,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in M. By using the fact that $\{\phi e_1,....,\phi e_{2n},\xi\}$ is also a local orthonormal basis, if we put $X = W = e_\alpha$ in (3.13) and sum up with respect to α, then

(3.14)

$$\sum_{\alpha=1}^{2n} g\big(R(\phi e_\alpha,\phi Y)\phi Z,\phi e_\alpha\big)=\frac{1}{2n-1}\sum_{\alpha=1}^{2n}\begin{bmatrix} g(Q\phi Y,\phi Z) \\ g(\phi e_\alpha,\phi e_\alpha)\end{bmatrix}$$
$$+g(\phi Y,\phi Z)g(Q\phi e_\alpha,\phi e_\alpha)-g(Q\phi e_\alpha,\phi Z)g(\phi Y,\phi e_\alpha)$$
$$-g(\phi e_\alpha,\phi Z)g(Q\phi Y,\phi e_\alpha)].$$

Now, using (3.4) – (3.8) the equation (3.14) takes the form

(3.15) $S(\phi Y,\phi Z)=(\tau-(f_1-f_3))g(\phi Y,\phi Z)$ and hence applying (2.1) and (2.10) into (3.15) we have

(3.16) $$S(Y,Z)=(\tau-(f_1-f_3))g(Y,Z)+(-\tau+(2n+1)(f_1-f_3))\eta(Y)\eta(Z),$$

which implies, M is an η-Einstein manifold. Now, by contracting (3.16) we obtain $(2n-1)\tau = 0$, which implies the scalar curvature $\tau = 0$. This completes the proof of theorem.

Definition 3([8]) *A differentiable manifold (M, g), satisfying the condition*

(3.17) $\phi^2 P(\phi X,\phi Y)\phi Z = 0$ *is called ϕ-projectively flat*.

Theorem 3 *Let $M(f_1, f_2, f_3)$ be a (2n+1)-dimensional generalized Sasakian space form, which is ϕ-projectively flat. If M is contact metric manifold with $\phi Q = Q\phi$, then it is η-Einstein manifold.*

Proof: Let M be a $(2n+1)$-dimensional, ϕ-projectively flat generalized Sasakian space form then, it is easily seen that $\phi^2 P(\phi X,\phi Y)\phi Z = 0$ holds if and only if

(3.18) $g(P(\phi X,\phi Y)\phi Z,\phi W)=0$, for any vector fields $X,Y,Z,W\in\chi(M)$. Using (2.14) ϕ-projectively flat means

(3.19) $$g(R(\phi X,\phi Y)\phi Z,\phi W)$$

$$=\frac{1}{2n-1}\left[g(\phi Y,\phi Z)g(Q\phi X,\phi W)-g(\phi X,\phi Z)g(Q\phi Y,\phi W)\right]$$

Similar to the proof of Theorem 1, we can suppose that $\{e_1,....,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in M. By using that $\{\phi e_1,....,\phi e_{2n},\xi\}$ is also a local orthonormal basis, if we put $X = W = e_\alpha$ in (3.19) and sum up with respect to α, then we have

(3.20) $$\sum_{\alpha=1}^{2n} g(R(\phi e_\alpha,\phi Y)\phi Z,\phi e_\alpha)$$

$$=\frac{1}{2n-1}\sum_{\alpha=1}^{2n}\begin{bmatrix} g(\phi Y,\phi Z)g(Q\phi e_\alpha,\phi e_\alpha)- \\ g(\phi e_\alpha,\phi Z)g(Q\phi Y,\phi e_\alpha)\end{bmatrix} z$$

Now, using (3.4) – (3.8) the equation (3.20) takes the form

(3.21) $2n S(\phi Y,\phi Z)=(\tau-(f_1-f_3))\,g(\phi Y,\phi Z)$,

and hence applying (2.1) and (2.10) into (3.21) we have

(3.22) $$S(Y,Z)=\left(\frac{\tau}{2n}-\frac{1}{2n}(f_1-f_3)\right)g(Y,Z)$$

$$+\left(-\frac{\tau}{2n}+\frac{1}{2n}(f_1-f_3)+2n(f_1-f_3)\right)\eta(Y)\eta(Z)$$

By contracting (3.22) we obtain $(2n-1)(f_1-f_3)=0$, which implies $f_1 = f_3$ and hence M is an η-Einstein manifold.

REFERENCES

[1] Alegre P., Blair D.E. and Carriazo A., Generalized Sasakian space forms, Israel J. Math., **141** (2004), 157-183.

[2] Arslan K., Murathan C. and Özgür C., On contact manifolds satisfying certain curvature conditions, Proceedings of the Centennial 'G.Vranceanu' and the Annual Meeting of the Faculty of Mathematics (Bucharest, 2000). An Univ. Bucuresti Mat. Inform., 49(**2**) (2000), 17-26.

[3] Arslan K., Murathan C. and Özgür C., On ϕ-conformally flat contact metric manifolds, Balkan J. Geom. Appl. (BJGA), **5(2)** (2000), 1-7.

[4] Blair D. E., Contact manifolds in Riemannian Geometry, Lecture Notes in Math. Springer-Verlag, Berlin-Heidelberg-New-York, **509** (1976).

[5] Cabrerizo J.L., Fernandez L.M., Fernandez M. and Zhen G., The structure of a class of K-contact manifolds, Acta Math. Hungar.,82 (**4**) (1999), 331-340.

[6] Mishra R.S., Almost contact metric manifolds, Monograph 1, Tensor Society of India, 1991

[7] Özgür C., ϕ-conformally flat Lorentzian para-Sasakian manifolds, Radovi Mathematicki, **12** (2003), 99

[8] Un Kyu Kim, Conformally flat generalized Sasakian space forms and locally symmetric generalized Sasakian space forms, Note di Matematica **26**, no.1, (2006), 55-67.

[9] Zhen G., On conformal symmetric K-contact manifolds, Chinese Quart. J. of Math, **7** (1992), 5-10.

[10] Zhen G., Cabrerizo J.L., Fernandez L.M. and Fernandez M., On ξ-conformally flat contact metric manifolds, Indian J. Pure Appl. Math, 28 (1997), 725-734.

[11] Yano K., Kon M., Structures on manifolds, Word Scientific, 1984.