A Note on G_a Partitions of n

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Abstract: **Andrews derived generating function for the number of smallest parts of *partitions* of positive integer *n*. Hanumareddy and Manjusri [5] derived generating function for the number of smallest parts of $partitions$ of n by using $r-partitions$ of n. In this chapter we defined the partitions of n with smallest parts of the form a^{k-1} where a, k are natural numbers, defined as $G_a\, partitions$ of n. In this chapter we derive generating function for $G_a spt(n)$ by using $r-G_a\, partitions$ of n. We also derive generating function for $sumG_a spt(n)$.**

Keywords: **Partition, r-partition, $G_a\, partition$, Smallest part of the $G_a\, partition$.**

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I. INTRODUCTION

Let $G_a\xi(n)$ be denote the set of all *partitions* of n with smallest parts are of the form a^{k-1}. Let $G_a p(n)$ be the cardinality of $G_a\xi(n)$ for $n \in N$ and $G_a p(0) = 1$. If $1 \le r \le n$, write $G_a p_r(n)$ for the number of *partitions* of n in $G_a\xi(n)$ each consisting of exactly r parts, i.e $r-partitions$ of n in $G_a\xi(n)$. If $r \le 0$ or $r \ge n$, we write $G_a p_r(n) = 0$. Let $G_a p(k,n)$ represent the number of *partitions* of n in $G_a\xi(n)$ using natural numbers at least as large as k only. Let the *partitions* in $G_a\xi(n)$ be denoted by $G_a\, partitions.$

Let $G_a spt(n)$ be denotes the number of smallest parts including repetitions in all *partitions* of n in $G_a\xi(n)$ and $sumG_a spt(n)$ be denotes the sum of the smallest parts. For $i \ge 1$, let us adopt the following notation on the lines of [3].

$G_a m_s(\lambda) =$ number of smallest parts of λ in $G_a\xi(n)$.

$$G_a spt(n) = \sum_{\lambda\in\xi(n)} G_a m_s(\lambda)$$

For example : $G_a\xi(8)$: $\qquad G_a p(8) = 21 \qquad G_a spt(8) = 56$

$\underline{8}, 7+\underline{1}, 6+\underline{2}, \underline{4}+\underline{4}, 6+\underline{1}+\underline{1}, 5+2+\underline{1}, 4+3+\underline{1}, 4+\underline{2}+\underline{2}, 3+3+\underline{2}, 5+\underline{1}+\underline{1}+\underline{1},$
$4+2+\underline{1}+\underline{1}, 3+3+\underline{1}+\underline{1}, 3+2+2+\underline{1}, \underline{2}+\underline{2}+\underline{2}+\underline{2}, 4+\underline{1}+\underline{1}+\underline{1}+\underline{1}, 3+2+\underline{1}+\underline{1}+\underline{1},$
$2+2+2+\underline{1}+\underline{1}, 3+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}, 2+2+\underline{1}+\underline{1}+\underline{1}+\underline{1}, 2+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}, \underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}.$

Existing generating functions are given below

Function	Generating function
$p_r(n)$	$\dfrac{q^r}{(q)_r}$
$p_r(n-k)$	$\dfrac{q^{r+k}}{(q)_r}$

number of divisors of the form a^{k-1} $\quad\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}$

sum of divisors of the form a^{k-1} $\quad\sum_{b=1}^{\infty}\frac{a^{k-1}q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}$ (1.1)

where $(q)_k=\prod_{n=1}^{k}(1-q^n)$ for $k>0$, $(q)_k=1$ for $k=0$ and $(q)_k=0$ for $k<0$.

and $(a)_n=(a;q)_n=(1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1})$ [2]

II. GENERATING FUNCTION FOR $G_a spt(n)$

The generating function for the number of smallest parts of all partitions of positive integer n is derived by Andrews. By utilizing $r-partitions$ of n, we propose a formula for finding the number of smallest parts of the form a^{k-1}.

2.1 Theorem:

$$G_a spt(n)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}p\left(a^{k-1},n-t.a^{k-1}\right)+\left\{d(n)\,|\,\text{divisors of form } a^{k-1}\right\}$$

Proof: Let $n=(\lambda_1,\lambda_2,\dots,\lambda_r)=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}},\left(a^{k-1}\right)^{\alpha_l}\right)$ be any $r-G_a\,partition$ of n with l distinct parts and $\mu_i=a^{i-1}\forall i$.

Case 1: Let $r>\alpha_l=t$ which implies $\lambda_{r-t}>a^{k-1}$

Subtract all a^{k-1}'s, we get $n-t.a^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}}\right)$

Hence $n-t.a^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t)-G_a\,partition$ of $n-t.a^{k-1}$ with $l-1$ distinct parts and each part is greater than or equal to $a^{k-1}+1$.

Case 2: Let $r>\alpha_l>t$ which implies $\lambda_{r-t}=a^{k-1}$

Omit a^{k-1}'s from last t places, we get $n-t.a^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}},\left(a^{k-1}\right)^{\alpha_l-t}\right)$

Hence $n-t.a^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}},\left(a^{k-1}\right)^{\alpha_l-t}\right)$ is a $(r-t)-partition$ of $n-t.a^{k-1}$ with l distinct parts and the least part is a^{k-1}.

Now we get the number of $r-G_a\,partitions$ with smallest part a^{k-1} that occurs more than t times among all $r-G_a\,partitions$ of n is $f_{r-t}\left(a^{k-1}, n-t.a^{k-1}\right)$.

Case 3: Let $r=\alpha_l=t$ which implies all parts in the *partition* are equal.

The number of $G_a\,partitions$ of n with equal parts is equal to the number of divisors of n which are of form a^{k-1}. Since the number of divisors of n which are of form a^{k-1} is $\left\{d\left(n\right) \mid \text{which are of form } a^{k-1}\right\}$ the number of $G_a\,partitions$ of n with all parts are equal and is of form a^{k-1} is $\left\{d\left(n\right) \mid \text{divisors of form } a^{k-1}\right\}$ where $\beta=\begin{cases}1 & \text{if } \dfrac{n}{r}=a^{k-1}\\ 0 & \text{otherwise}\end{cases}$.

From cases (1), (2) and (3) we get $r-G_a\,partitions$ of n with smallest part a^{k-1} that occurs t times is

$$f_{r-t}\left(a^{k-1}, n-t.a^{k-1}\right)+p_{r-t}\left(a^{k-1}+1, n-t.a^{k-1}\right)+\beta$$

$$=p_{r-t}\left(a^{k-1}, n-t.a^{k-1}\right)+\beta \quad \text{where} \quad \beta=\begin{cases}1 & \text{if } \dfrac{n}{r}=a^{k-1}\\ 0 & \text{otherwise}\end{cases}$$

Hence the number of smallest parts in $G_a\,partitions$ of n is

$$G_a spt\left(n\right)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}p\left(a^{k-1}, n-t.a^{k-1}\right)+\left\{d\left(n\right) \mid \text{divisors of form } a^{k-1}\right\} \blacksquare$$

2.2. Theorem: $p_r\left(a^{k-1}+1, n\right)=p_r\left(n-a^{k-1}.r\right)$

Proof: Let $n=(\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_i > a^{k-1}\ \forall i$ be any $r-partition$ of n.

Subtracting a^{k-1} from each part, we get $n-a^{k-1}.r=\left(\lambda_1-a^{k-1}, \lambda_2-a^{k-1}, \ldots, \lambda_r-a^{k-1}\right)$

Hence $n-a^{k-1}.r=\left(\lambda_1-a^{k-1}, \lambda_2-a^{k-1}, \ldots, \lambda_r-a^{k-1}\right)$ is a $r-G_a\,partition$ of $n-a^{k-1}.r$

Therefore the number of $r-G_a\,partitions$ of n with parts greater than or equal to $a^{k-1}+1$ is $p_r\left(n-a^{k-1}.r\right)$. $\blacksquare$

2.3. Theorem: $\displaystyle\sum_{n=0}^{\infty}G_a spt\left(n\right)q^n=\frac{1}{\left(q\right)_\infty}\sum_{n=1}^{\infty}\frac{q^n\left(q\right)_{n-1}}{\left(1-q^n\right)}$.

Proof: From theorem (2.1) we have

$$G_a spt(n)=\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}p\left(a^{k-1},n-t.a^{k-1}\right)+\left\{d(n)\,|\,\text{divisors of form } a^{k-1}\right\}$$

first replace $a^{k-1}+1$ by a^{k-1} then replace n by $n-t.a^{k-1}$ in theorem $(2.2.)$ where

$$=\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{k=1}^{\infty}p_r\left(n-t.a^{k-1}-r\left(a^{k-1}-1\right)\right)+\left\{d(n)\,|\,\text{divisors of form } a^{k-1}\right\}$$

$\left\{d(n)\,|\,\text{divisors of form } a^{k-1}\right\}$ is the number of positive divisors of n which are form a^{k-1}.

From (1.1)

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+t.a^{k-1}+r\left(a^{k-1}-1\right)}}{(q)_r}+\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{t.a^{k-1}+r.a^{k-1}}}{(q)_r}+\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{t.a^{k-1}}\left[\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^r}{(q)_r}\right]+\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left[\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^r}{(q)_r}\right)-1\right]+\sum_{r=1}^{\infty}\frac{q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^r}{(q)_r}\right)$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^r q^{a^{k-1}}}\right)\qquad\text{from [2]}$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^{r+a^{k-1}}}\right)$$

$$=\frac{1}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}(q)_{a^{k-1}-1}}{\left(1-q^{a^{k-1}}\right)}$$

$$=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^{a^{k-1}}(q)_{a^{k-1}-1}}{\left(1-q^{a^{k-1}}\right)}\ \blacksquare$$

2.4. Corollary: The generating function for $A_c(n)$, the number of smallest parts of the $G_a\, partitions$ of n which are multiples of c is

$$\sum_{n=0}^{\infty} A_c(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{q^{c.a^{n-1}} (q)_{c.a^{n-1}-1}}{\left(1-q^{c.a^{n-1}}\right)}$$

2.5. Theorem: The generating function for the sum of smallest parts of the $G_a\, partitions$ of n is

$$\sum_{n=0}^{\infty} sum\, G_a spt(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{a^{n-1} q^{a^{n-1}} (q)_{a^{n-1}-1}}{\left(1-q^{a^{n-1}}\right)}$$

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