Level Crossings of a Random Trigonometric Polynomial with Dependent Coefficients

Dipty Rani Dhal[1], Dr. P. K. Mishra[2]

[1]*Research Scholar, SOA University, BBSR, Odisha, India*

[2]*Associate Professor of Mathematics, CET, BPUT, BBSR, Odisha, India*

***Abstract:-* Let $P_n(\theta) = j^p g_i(w)\cos j\theta$ be a random trigonometric polynomial such that the coefficients $\{g_1(w), g_2(w)\ldots g_n(w)\}$ is a sequence of normally distributed independent random variables with mean zero and variance one and the correlation coefficients f_{ij} between ith and jth coefficients are constant. 0<f<1. We have to find the average number of real zeros $E_n(0,2\pi)$ of the equation $T_n(\theta) = K$ (where K is any constant).**

I. INTRODUCTION

Let $$T_n(\theta.\omega) = \sum_{j=1}^{n} j^p g_i(w)\cos j\theta \qquad (1)$$

Be a random trigonometric polynomial, where the coefficients $\{g_1(w). g_2(w)\ldots g_n(w)\}$ (0, w-1) is a sequence of independent and normally distributed random variables with mathematical expectation zero and variance one.

Let $ENn(\alpha,\beta)$ be the number of level crossings of the family of the curves $Y = Tn = (\theta) = 0$ with the line y=k for α≤θ≤β. Previously Dunnage [3] found that in the case of normally distributed random variables with mean zero and variance one and p=0 the polynomial (1.) has $\left(\frac{2n}{3}\right) + 0\left(n^{\frac{11}{13}}(\log n^{\frac{3}{13}})\right)$ zeros on the average in the interval (0,2π) except for a certain exceptional set whose measure does not exceed $(\log n)^{-1}$. Nayak N.N. and Patanayak (6) considered the same polynomial and found that the average number of crossings of these curves with the line y=k is asymptotic to $\left(\frac{2n}{3}\right)$ for $K \neq 0(n)$.

II. THEOREM

Let $P_n(\theta) = j^p g_i(w)\cos j\theta$ be a random trigonometric polynomial such that the coefficients $\{g_1(w), g_2(w)\ldots g_n(w)\}$ is a sequence of normally distributed independent random variables with mean zero and variance one and the correlation coefficients f_{ij} between ith and jth coefficients are constant. 0<f<1. Then for sufficiently large n and p >0, the average number of real zeros of the equation $T_n(\theta) = K$ (where K is any constant) satisfies

$$E_n(0,2\pi) = 2n\left(\frac{2p+1}{2p+3}\right)^{1/2} + 0(n^{3/4}) \text{ if } K = 0\,(n^{3/8})$$

$$E_n(0,2\pi) = 2n\left(\frac{2p+1}{2p+3}\right)^{1/2} + 0(n) \text{ if } K = 0\,(n)$$

Das, M.K (1) and Sambadham, M. and Renganathan (8) separately used Kac Rice formula to find the mathematical expectation of the number of real roots of (1) for P=0 and K=0 and for some non zero finite or infinite value of mean. In this wider class of distribution he found the same expected number of real roots for said polynomial when ever the correlation coefficients between any two coefficients g_j and g_r denoted by f_{jr} is constant 0<f<1 as have been found in previously mentioned works.

Samal and Pratihari (7) considered same polynomial when mean zero and variance one and f_{ij} the correlation coefficients between j th and ith coefficients are constant and for any constant K the expected number of real roots of the equation T(θ)=K satisfies

$$EN(0,2\pi) = \left(\frac{2n}{3}\right) + 0(n^{3/4}) \text{ if } K = 0\,(n^{3/8})$$

$$EN(0,2\pi) = \left(\frac{2n}{3}\right) + 0(n) \text{ if } K = 0\,(n)$$

Here we consider a polynomial of the form Let

$$T_n(\theta) = T_n(\theta,\omega) = \sum_{j=1}^{n} j^p g_j(w)\cos j\theta = K$$

Where the random variables are normally distributed with mean zero and unity variance. We denote EN (a,b). The average number of level crossings of the family of curves $y=T_n(\theta)$ with the line y=k.

III. KAC RICE FORMULA FOR TRIGONEMETRIC POLYNOMIAL

From Kac Rice formula we obtain

$$EN_n(\alpha,\beta)=\int_{\alpha}^{\beta} d\theta \int_{-\infty}^{\infty} |y|(0,y)dy \qquad (2)$$

where (x_1,x_2) is the density of the joint distribution function $T_n(\theta)$ and $T'_n(\theta)$ Let $(t)=\frac{1}{\sqrt{2\pi}}\int_0^t \exp\left(\frac{-y^2}{2}\right)dy$

and $(t)=(t)\frac{1}{\sqrt{2\pi}}\exp\left(\frac{t^2}{2}\right)dy$

Using the procedure to find out the number of level crossings used by Cramer and Lead better for the equation $T_n(\theta)=K$ we obtain

$$EN_n(a,b)=\int_a^b\left(\frac{}{X}\right)^{1/2}(1-\lambda)^{1/2}.\left(\frac{\xi}{X}\right)\left[20(\mathrm{n})+\mathrm{n}\{20(\mathrm{n})-1\}\right]$$

where

$$\alpha=Var\{\mathrm{T_n}(0)-\mathrm{K}\}$$
$$\beta=Var\{\mathrm{T_n}(0)\}$$
$$v=(XZ)^{1/2}COV\left[\{\mathrm{Tn}(\theta)-\mathrm{K}\}.Tn(\theta)\right]$$
$$\xi=E\{T_n(\theta)-K\}$$
$$n=(Z)^{-1/2}(1-\lambda)^{-1/2}\left[v-\frac{\lambda\xi v\mathrm{z}}{x}\right]$$
$$and\ \mathrm{v}=\mathrm{E}\{\mathrm{T_n}(\theta)\}$$

Since the coefficients of $T_n(\theta)$ are independent and normally distributed random variables with mean zero and variance one

We can easily derive that

$$\xi=\sum_{j=1}^{n} j^p\cos j\theta-K=m_1-K$$

$$v=\sum_{j=1}^{n} j^{p+1}Sinj\theta=m_2$$

$$X=\sum_{j=1}^{n} j^{2p}Cos^2\mathrm{j}\theta$$

$$Z=\sum_{j=1}^{n} j^{2p+2}Sin^2\mathrm{j}\theta$$

$$\lambda=\frac{-Y}{XZ}$$

$$n=\frac{Y(m_1-K)}{VAX}$$

and $\Delta=XZ-Y^2$

Since $\Phi(t)=\frac{1}{2}+\frac{1}{v\pi}erf\left(\frac{1}{2}\right)$ and using (2) we have the extended Kac Rice formula.

$$E_n(a,b)=\int_b^a\frac{v\Delta}{\pi\alpha}\exp\frac{-k^2\beta}{2\Delta}d\theta+\sqrt{\frac{2}{\pi}}\int_b^a|kv|\alpha^{-\frac{3}{2}}\exp\left(\frac{k^2}{2\alpha}\right)erf\left(\frac{-kv}{2\alpha\beta}\right)d\theta$$

$$=\mathrm{I_1(a,b)+I_2\ (a,b)}$$

Where

$$\Delta=\alpha\beta-v^2$$

$$A^2=\sum_{j=1}^{n} j^{2p}Cos^2 j\theta$$

$$B^2=\sum_{j=1}^{n} j^{2p+2}Sin^2 j\theta$$

$$C=\sum_{j=1}^{n} j^{2p}Cosj\theta$$

$$E=\sum_{j=1}^{n} j^{p+1}Sinj\theta$$

Proof of the Theorem

We divide the zeros of $T_n(\theta)$ into two groups and proceed to estimate expected number of zero in each group.

The zeros are

(i). those lying in the intervals (0,ε), (π-ε, π+ε) and (2π-ε, 2π)
(ii). Those lying in the intervals (ε, π-ε) and (π+ε, 2π- ε)

It so happens that zeros in group (i) intervals contribute insignificantly towards $EN_n(0,2\pi)$ unlike those in group (ii) intervals we establish this in the next two sections together with the proof theorem. We choose ε=n-1/4

IV. NUMBER OF ZEROS IN GROUP (i) INTERVALS

We know that out side a small exceptional set of values of ω, $T_n(\theta)$ has a negligible number of zeros in the intervals (0, ε), (π-ε, π+ε) and (2π-ε, 2π). By periodically the number of zeros in (0, ε) and (2π-ε, 2π) is same as the number of zeros in (0, ε) we prove out assertion for the interval (-ε, ε) first. The estimation of roots in the interval (π-ε, π+ε) follows the same lines of argument.

Let $\mathrm{T_\omega(Z)-K=T_n-K=\sum_{j=1}^{n}g_1(\omega\omega^p)cosjZ}$

The distribution function of the random variable

$\mathrm{T_\omega(0)=\sum_{j=1}^{n}j^p g_j(\omega\alpha}$ is given by

$$G(X) = \frac{1}{2\pi}\int_0^x e\frac{t^2}{2^2}dt$$

Where

$$X^2 = (1,f)\sum_{j=1}^{n} j^{2p+1}\left(\sum_{j=1}^{n} jp\right)^2 = (1-f)n^{2p+1} + fn^{2p+2}$$

Hence

$$P\{-e^{-2ne}T_{\omega}(o) - K.-e^{2n}\}$$

From which any positive we can see that T(0,w)-K>e^{-v} except for sample functions from v-set measure not exceeding

$$f\left|T(2\varepsilon\, e^{i\theta})\right| < 2\varepsilon^{2ne}\left(|g_1| + |g_2| + ...|g_n|\right) \le 2n\varepsilon^{2ne}\max|g_j| \quad (3)$$

Hence

$$\left|T(2\varepsilon\, e^{i\theta})\right| < 2\varepsilon^{2ne}\left(1^p + 2^p + 3^p + ...n^p\right)\max|g_j|$$

Hence $g_1=g_1(w)$

If $\max\limits_{1<j<n}|g_j| > n \text{ then } |g_j| > n$ for as least are values of j and j≤n from the distribution function of random variable g_j we have

$$\left|T(2\varepsilon\, e^{i\theta})\right| < 2\varepsilon^{2ne}Dn\left(\max_{1<j<n}\right)g_j\ (w)$$

where $D_n=1^p+2^p+.......+n^p$

Hence $|T(0,\omega) - K| > 2\varepsilon^{2ne}$ except for a set of measure at most e^{-ne}

Now if

$$\max|g_j| > ne^v then |g_j| > ne^v \text{ for at least one value of } j \le n \text{ so that}$$

Prob

$$(\max)\ |g_j| > ne^v \le \sum_{j=1}^{n} prob|g_j| > ne^v \le n\ prob|g_j| > ne^v$$

$$= \sqrt{\frac{2}{\pi}}\int_v^0 \exp\left(\frac{-t^2}{2}\right)dt \approx \sqrt{\frac{2}{\pi}}\exp\left(-v - n^2\varepsilon^{2v/2}\right) \quad (4)$$

For sufficient large n

Therefore from (3) and (4) except for sample in an w set of measure not exceeding $\left(\frac{2}{\pi}\right)^{1/2}\exp\left(-v - n^2\varepsilon^{2v/2}\right)$

$$\left|T(2\varepsilon\, e^{i\theta})K\right| < n^2\exp(2ne+v) - K$$

$$\left|T(2\varepsilon\, e^{i\theta}w)\right| \le 2e^{2ne}\left|\max_{1<j<n}\right| j^p g_j(w) \le 2\varepsilon^{2n}D_n\left(\max_{1<j<n}|g_j(w)|\right) \quad (5)$$

From the distribution function g_j we have

$$P\left(\max_{1<j<n}|g_j > n|\right) < \sqrt{\frac{2}{\pi}\int_n^0\left(\frac{-t^2}{2}\right)dt}$$

$$< n\sqrt{\frac{2}{\pi}}\int_0^n \frac{e^{\frac{1}{2x2}}}{2}dx$$

$$= n\sqrt{\frac{2}{\pi}}\int_0^n \frac{e^{\frac{n}{2}}}{2}$$

Hence $\max\limits_{1<j<n}|g_j > n|$ except for a set of measure at most $e^{n2/3}$ for large n

Hence $T(2\in e^{i\theta}\omega < 2nD_n \in^{2ne+v}$ for some except for a set of measure $-n^2/3$

Combining (4) and (5) and since both K=0 ($n^{3/8}$) and K=0

$$\left|n^2\exp(2n+v) + K < 2n2\exp(2n+V)\right| \quad (6)$$

Let N(e) be the number of zero of T (z,w)-K =0 in Z ≤e using Jensen theorem

$$N(\in) \le \frac{1}{2\pi\log 2}\int_0^{2\pi}\log\left|\frac{T_m(2\in e^{i\theta}) - K}{T(0) - K}\right|d\theta \quad (7)$$

This holds good for every set $g_1(w)…g_n(w)$ with T_w-K=0

Since $T(2\in e^{i\theta}\omega K 2nD_n \in^{2ne+v}\ as\ K = 0$

Thus

$$\left|\frac{T_m(2\in e^{i\theta}) - K}{T(0) - K}\right| < 2_n D_n^{4ne}$$

except for a set of measure $e^{-2n/3}+e^{-ne}<2e^{-ne}$

$$N(\in)\log 2 < \frac{1}{2\pi}\int_0^{2\pi}\log D_n + 4^{ne}d\theta$$

Hence

$$or\ N(\in)\log 2 + (\log 2 + \log n + \log D_n + 4n)$$

$$\text{or } N(\in) < 1 + \frac{\log 2 + \log n + \log D_n + 4n}{\log 2}$$

From the above derivations it is easy to conclude that the number of zeros of T(z,w)-K in the region Z is 0 ($n^{3/4}$) except for a set of measure not exceeding e^{-ne}.

Excepted number of level crossing in Group (ii) intervals. We make the interval (ε,π- ε) and get the following estimates.

First we recall some of the estimates used in Das [8]. Let S_k=cos2θ+cos4 θ+……cos2k θ

For ε≤θ≤π we have

$$\frac{1}{\text{Sin} \in} < S_k < \frac{1}{\text{Sin} \in}$$

So that by Abel's theorem we have

$$\sum_{k=0}^{n} k^{2p}\operatorname{Cos} 2k + 1 = S_1(2^{2p} - 1) - S_2(3^{2p} - 2^{2p}) + S_{n-1}\{(n-1)^{2p} + (n-2)^{2p}\}$$

$$+ S_n n^{2p} = 0 - \frac{n^{2p}}{S_m}$$

By using the expected number of level crossings given by Cramer and lead better for the equation T(θ)-K we can obtain

$$EN(a,b)=\int_a^b \frac{N^{1/2}}{\theta}\left(\frac{-K}{a^{1/2}}\right)\left[\Phi\left(\frac{kv}{\alpha\Delta}\right)+Kv\,2\Phi\left(\frac{kv}{\sqrt{\alpha\Delta}}\right)^{-1}\right]d\theta$$

where

$$\alpha=Var\{\mathrm{T}(\theta)\}=\mathrm{A}^2+\sum_{\mathrm{j}} f_{jr}Cos_j\,\theta Cosr\theta$$

$$\beta=Var\{\mathrm{T}'(\theta)\}=B^2+\sum_{\mathrm{j}} f_{jr}Cos_j\,\theta Sinr\theta$$

$$v=Var\{\mathrm{T}'(\theta)\}=C+\sum_{\mathrm{j}} f_{jr}Cos_j\,\theta Sinr\theta$$

We recall some of the estimate used in Das (1) and also Abels theorem we have

$$A^2\sum_{j=1}^{n} j^{2p}Cos^2 j\theta=\frac{1}{2}\sum_{j=1}^{n} j^{2p}+\frac{1}{2}\sum_{j=1}^{n} j^{2p}Cos^2 j\theta$$

$$=\frac{\mathrm{n}^{2\mathrm{p}+1}}{2(2\mathrm{p}+1)}\left[1+0\left(\frac{1}{n}\right)\right]$$

$$B^2=\frac{\mathrm{n}^{2\mathrm{p}+3}}{2(2\mathrm{p}+3)}\left[1+0\left(\frac{1}{n}\right)\right]$$

$$C=0\left(\frac{\mathrm{n}^{2\mathrm{p}+1}}{2(2\mathrm{p}+3)}\right)$$

$$D=0\left(\frac{\mathrm{n}^{\mathrm{p}+1}}{2(2\mathrm{p}+3)}\right)$$

by taking $n^{3/4}$ and we consider the correlation coefficient f_{jr}=f=constant

$$\alpha=A^2+f\sum_{j}^{n}\cos j\theta\cos r\theta=\mathrm{A}^2+\left\{\sum_{j=1}^{n}(\cos j\theta)^2+\sum_{j=1}^{n}(\cos^2 j\theta)\right\}$$

$$=(1-f)A^2+fD^2$$

$$=(1-f)\frac{n^{2p+1}}{2(2p+1)}+0\left[\left(\frac{1}{n\in^2}\right)\right]$$

Similar

$$\beta=(1-t)\beta^2+tE^2=(1-f)\frac{n^{2p+3}}{2(2p+3)}+0\left[\left(\frac{1}{n\in^2}\right)\right]$$

$$and\ \mathrm{v}=(\mathrm{t}-1)\mathrm{C}+\mathrm{tDE}=(10\mathrm{t})0\left(2\mathrm{p}+1\big/2\right)$$

$$and\ \mathrm{V}=\alpha\beta-\mathrm{v}^2=(1-t)^2\left[\frac{n^{4p+4}}{4(2p+1)(2p+3)}\right]0n^3\in^{-2}$$

Hence

$$\frac{\Delta^{1/2}}{\alpha}=\left(\frac{2p+1}{2p+3}\right)^{1/2}n[1+0(\in)]]$$

Let

K=0(vn)

Then after some simplification

$$\frac{\Delta^{1/2}}{\alpha}=\left[\frac{k\beta}{2N}\right]=\exp\left[\frac{k}{n^{2p+1}}+0\left(\frac{1}{n}\right)\right]$$

$$\frac{|vk|}{\alpha^{3/2}}=0\left(\frac{n^{1/2}}{}\right)$$

and $\exp\left(\frac{k^2}{2\alpha}\right)=\exp\left(\frac{-\mathrm{k}^2}{\mathrm{n}^{2\mathrm{p}+1}}+0\left(\frac{1}{n\in^2}\right)\right)$

From the above estimates we have

$$\int_{\in}^{\pi-\in} p_1(\theta)d\theta=\int_{\in}^{\pi-\in}\frac{\sqrt{N}}{\pi\theta}\exp\left(\frac{-k2\beta}{n\in^2}\right)d\theta$$

$$=\left(\frac{2\mathrm{p}+1}{2\mathrm{p}+3}\right)n[1+0(\theta)]=\exp\left[\frac{-k2}{n2p+1(1-f)}+0\left(\frac{1}{n\in^2}\right)\right]$$

$$=\int_{\in}^{\pi-\in} p_1(\theta)d\theta=\int_{\in}^{\pi-\in}\frac{\sqrt{N}}{\pi\theta}\exp\left(\frac{-k2}{2\alpha}\right)erf\left(\frac{-kv}{2\alpha\Delta}\right)d\theta$$

$$=0\left(\frac{\mathrm{n}}{\mathrm{p}}\right)\exp\left\{\left(\frac{-k^2}{(1-f)n^{2p+1}}\right)+0\left(\frac{1}{n\in^2}\right)\right\}$$

for large n

$$\exp\left\{\frac{-k2}{n2p+1(1-f)}+0\left(\frac{1}{n\in^2}\right)\right\}\to 1$$

$$E_n(\in,\pi-\in)=\int_e^a t_1(\theta)d\theta=+\int_e^a t_1(\theta)d\theta$$

$$=\left(\frac{2\mathrm{p}+1}{2\mathrm{p}+3}\right)+0(vn)$$

$$When\ K=0\,(\mathrm{n}^{3/8})$$

$$E_n(\in,\pi-\in)=\left(\frac{2\mathrm{p}+1}{2\mathrm{p}+3}\right)+0(n^{3/4})$$

A similar estimate can be derived for $EN_n(\pi+\in,\pi-\in)$

We have

$$E_n(\in,\pi-\in)=\left(\frac{2\mathrm{p}+1}{2\mathrm{p}+3}\right)^{1/2}+0(Nn)\ \text{when K=0 (n)}$$

$$E_n(\in,\pi-\in)=\left(\frac{2\mathrm{p}+1}{2\mathrm{p}+3}\right)^{1/2}+0(n^{1/4})\ \text{when K=0 }(\mathrm{n}^{3/8})$$

which complements the proof of the theorem

REFERENCES

[1]. **Das, M.K.** Real zeros of a random sum of orthogonal polynomials, ***Proc. Amer. Math. Soc. 27, 1 (1971), 147-153.***

[2]. **Das, M.K.** Real zeros of a class of random algebraic Polynomials, ***Journal of Indian Math. Soc.. 36 (1972), 53-63.***

[3]. **Dunnage, J.E.A.** The number of real zeros of a class of random algebraic polynomials (I), ***Proc. London Math. Soc. (3) 18 (1968), 439-460.***

[4]. **Nayak N.N. and Patanayak, S**. Strong results for real zeros of random polynomial, **Pac.** ***Jour Math. 103 (1982), 509-522.***

[5]. **Pratihari and Bhanja .**On the number of real zeros of random trigonometric polynomial, ***Trans. Of Amer. Math. Soc. 238 (1978) 57-70.***

[6]. **Samal and Pratihari** . Real zeros of a random algebraic polynomial, **Quar.** ***Jour. Math. Oxford. 2 (1993), 169-175.***

[7]. **Samal and Pratihari** . The number of real zeros of a class of random algebraic polynomials (I), ***Proc. London Math. Soc. (3) 18 (1989), 439-460.***

[8]. **Sambadham, M. and Renganathan**. On the number of real zeros of a random trigonometric polynomial, coefficient with non-zero mean, ***Jour. Indian. Math. Soc. 45 (1989), 193-203.***