BEM Solution for the Radiation BC Thermal Problem with Adaptive Basis Functions

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Abstract: **In this work, heat transfer problems that have Radiation boundary condition are addressed with a unique BEM procedure. To address this, adaptive shape functions are defined on the nodes in contrast to the standard BEM procedure. The shape function are expandable to solve the complex mathematical problems that arise in the solution of the equations. The solution developed using the adaptive node shape functions are compared with that of the conventional node basis shape function. The shape functions yield comparable results with conventional node basis shape function by reducing the computational time. Results are plotted for several mesh sizes and the convergence study is also made. Effort is made to improve the accuracy of the solution. Finally, important conclusions are drawn and future scope is defined.**

Keywords: **Boundary Element Methods, Heat transfer, Node Based Basis Functions, Adaptive Basis Functions and Error Analysis.**

I INTRODUCTION

The Boundary Element Method [BEM] or the Boundary Integral Equations [BIE] are popularly used to solve the exterior or open region problems due to their capability of encompassing the radiation boundary condition into Green's function. The problem with the exterior problem is, in order to impose the radiation boundary condition on the boundary of the problem domain, the space surrounding the object need to be discretized as large as possible. The solution time becomes enormously high as the number of discretization elements increases. Therefore it becomes difficult to solve such problems as the quantity of the elements grows.

Problems can be solved using BEM by discretizing only the surface of the object instead of discretizing the whole space around the object. For the surface of the object, boundary conditions such as constant temperature boundary conditions, heat flux boundary conditions etc. can be applied and the radiation boundary conditions is not required, as the problem formulation itself takes care of it in the form of the Green's function.

Selection of shape function play a critical role in obtaining the accurate solution. In the BEM or BIE procedure, the shape functions are referred as basis functions. One has flexibility to define the basis functions on any of the geometrical entities. Several geometric entities are formed when a surface of an object is discretized into triangles. These entities are triangular patches or faces, edges and nodes. The triangular patch modeling for a closed surface results in certain number of geometric entities. If N_e is the number of edges created on the closed surface of the object, then it will have the number of faces equal to $2N_e/3$ and number of nodes equal to $(N_e/3) +2$. That means,

$$N_f = 2N_e/3 \text{ and} \quad (1)$$

$$N_n = (N_e/3) + 2 \quad (2)$$

The BEM solution offers flexibility so that one can define the basis function on any of the geometric entity type. Edge, face and node type's basis functions are the basis functions defined on edge, face and node respectively. After solving the integral equations numerically, the resulting matrices are in the sizes of $N_n \times N_n$, $N_f \times N_f$, and $N_e \times N_e$ respectively for the node type, face type and edge type basis functions. For example, an object of surface discretized into 6000 edges, then it results in 4000 faces and 2002 nodes. That means matrices are 6000 × 6000 for edge type, 4000 × 4000 for face type and 2002 × 2002 for node type basis functions. Therefore computational complexity of solving edge type and face type basis function is very high as compared to solving it using node type basis function.

In this research an attempt has been made to solve the thermal problem with another set of basis functions that is an extension of node type basis functions by Raghu Kumar at al. [1]. These new basis functions are called adaptive basis functions. These functions are not used in the solution of the thermal problems; but are widely used in electromagnetic and acoustic scattering problems. Adaptive basis function was first introduced by Chandrasekhar [3,4] and S.M Rao at al. [5,6] In this work, an attempt has been made to extend the adaptive node basis functions to solve the radiation boundary condition thermal problems.

II BEM PROCEDURE

The resulting matrix obtained after numerical solution of integral equations, has to be solved by any linear equation solvers when node type basis functions are used.
If the resulting linear equation is

$$\mathbf{ZX = Y}, \quad (3)$$

Then X can be found by

$$\mathbf{X = Z^{-1}Y} \qquad \mathbf{(4)}$$

Where iterative methods are used to obtain the inverse of the matrix **[Z]**

If the N_n is the size of the matrix **[Z],** then the complexity in solving the matrix **[Z]** is in the order of $\mathbf{N_n^3}$, which is big as compared to the size of the matrix. The BEM solution results in a full matrix instead of a sparse matrix or diagonal matrix.

In this article, attempt is made to generate the diagonal matrix of matrix **[Z]** by defining the basis functions on the nodes. But each of these basis functions are not defined on each node in isolation, but each basis function is defined on a set of nodes so that the resulting values in the matrix **[Z]** are always diagonal; and off diagonal elements are either zero or near zero.

By defining the basis functions spanning over several nodes and by assigning appropriate weights, the total effect on any other node can be reduced to zero. Thus producing a null field. This type of basis function is known as adaptive basis function.

III DEVELOPMENT OF ADAPTIVE BASIS FUNCTIONS

Consider a cluster of nodes and let adaptive basis function to be defined on main node n_i. Six nodes are surrounding the main node and have their own node type basis functions. Boundary type of adaptive basis function is indicated in red color in Fig 1. Let there are N basis functions in the cluster and each node type basis function associated with weight is designated as $\lambda_n, n = 1,2,3,\ldots\ldots, N$.

Let there are N_n nodes on the surface of the object that resulted from triangular patch modeling, and when choosing the number of basis functions in the cluster, there is no limit on the quantity, but it should be less than that of total number of node basis functions on the surface. Null field is produced when whole adaptive basis function is tested on any node outside the cluster. It is not necessary that all the nodes chosen for testing need to be chosen from outside the cluster. The testing node should not be the main node and it can be even well within the cluster.

The number of testing nodes to be chosen either equal to or greater than the number of nodes in the cluster in order to find the weights in the cluster. If the number of nodes in the cluster is equal to number of testing nodes, it results in exact solution for weights. It the number of testing nodes is greater than cluster nodes, it results in an over determined system of linear equations and one gets appropriate values for the weights that satisfies all the linear equations.

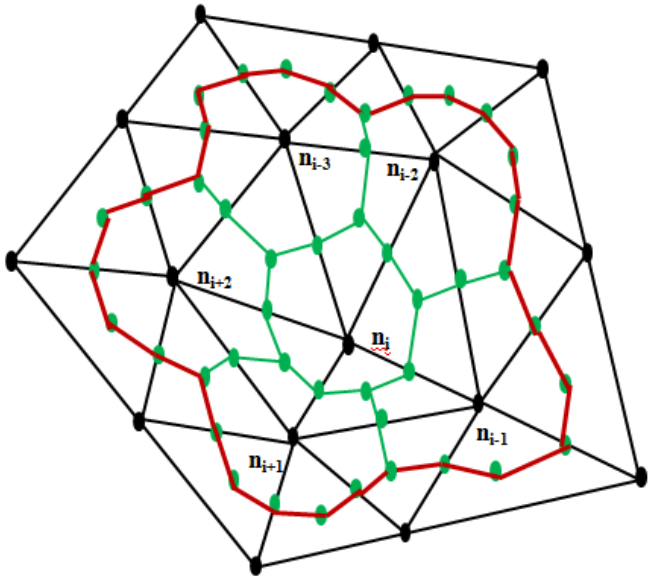

Figure 1: Adaptive Basis Function on a Node

Weights can be evaluated using the below procedure:

Let main node is surrounded by N number of nodes in the cluster, then total N+1 number of nodes are there in the cluster. Let there be N weights associated to each of the surrounding nodes designated by $\lambda_n, n = 1,2,3,\ldots\ldots, N$ which are to be determined. Number of testing points to be at least N, since there are N number of weights to be determined. Let N=6, then nodes are designated as $n_{i-3}, n_{i-2}, n_{i-1}, n_i, n_{i+1} and\ n_{i+2}$.

The designations given above are arbitrary. When tested on the number of nodes N_n, it results in the following set of linear equations.

$$B_{1,i-3}\lambda_{i,i-3} + B_{1,i-2}\lambda_{i,i-2} + B_{1,i-1}\lambda_{i,i-1} + B_{1,i} + B_{1,i+1}\lambda_{i,i+1} + B_{1,i+1}\lambda_{i,i+1} = 0$$

$$B_{2,i-3}\lambda_{i,i-3} + B_{2,i-2}\lambda_{i,i-2} + B_{2,i-1}\lambda_{i,i-1} + B_{2,i} + B_{2,i+1}\lambda_{i,i+1} + B_{2,i+1}\lambda_{i,i+1} = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$B_{Nn,i-3}\lambda_{i,i-3} + B_{Nn,i-2}\lambda_{i,i-2} + B_{Nn,i-1}\lambda_{i,i-1} + B_{Nn,i} + B_{Nn,i+1}\lambda_{i,i+1} + B_{Nn,i+1}\lambda_{i,i+1} = 0$$

$$B_{1,i-3}\lambda_{i,i-3} + B_{1,i-2}\lambda_{i,i-2} + B_{1,i-1}\lambda_{i,i-1} + B_{1,i+1}\lambda_{i,i+1} + B_{1,i+1}\lambda_{i,i+1} = -B_{1,i}$$

$$B_{2,i-3}\lambda_{i,i-3} + B_{2,i-2}\lambda_{i,i-2} + B_{2,i-1}\lambda_{i,i-1} + B_{2,i+1}\lambda_{i,i+1} + B_{2,i+1}\lambda_{i,i+1} = -B_{2,i}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$B_{Nn,i-3}\lambda_{i,i-3} + B_{Nn,i-2}\lambda_{i,i-2} + B_{Nn,i-1}\lambda_{i,i-1} + B_{Nn,i+1}\lambda_{i,i+1} + B_{Nn,i+1}\lambda_{i,i+1} = -B_{Nn,i}$$

This can be expressed in the matrix form as

$$[B]\{\Lambda\} = \{b\} \qquad (5)$$

Where,

$$B = \begin{bmatrix} B_{1,i-3} & B_{1,i-2} & ..... & B_{1,i+2} \\ B_{2,i-3} & B_{2,i-2} & ..... & B_{2,i+2} \\ ..... & ..... & ..... & ..... \\ B_{Nn,i-3} & B_{Nn,i-2} & ..... & B_{Nn,i+2} \end{bmatrix} \quad (6)$$

$$\Lambda = \begin{Bmatrix} \lambda_{i,i-3} \\ \lambda_{i,i-2} \\ ..... \\ \lambda_{i,i+2} \end{Bmatrix} \quad (7)$$

$$b = \begin{Bmatrix} B_{1,i} \\ B_{2,i} \\ ..... \\ B_{Nn,i} \end{Bmatrix}$$

In other words,

$$\begin{bmatrix} B_{1,i-3} & B_{1,i-2} & ..... & B_{1,i+2} \\ B_{2,i-3} & B_{2,i-2} & ..... & B_{2,i+2} \\ ..... & ..... & ..... & ..... \\ B_{Nn,i-3} & B_{Nn,i-2} & ..... & B_{Nn,i+2} \end{bmatrix} \begin{Bmatrix} \lambda_{i,i-3} \\ \lambda_{i,i-2} \\ ..... \\ \lambda_{i,i+2} \end{Bmatrix} = \begin{Bmatrix} B_{1,i} \\ B_{2,i} \\ ..... \\ B_{Nn,i} \end{Bmatrix} \quad (8)$$

The Eq. 8 need to be solved to get the values of the weights. The elements of **[B]** and {b} can be computed using the normal node type basis functions. The equation is an over determined system of linear equations and least square technique can be used to calculate the weights vector {Λ} After calculating the weights $\lambda_{i,n}$, the i^{th} basis function can be constructed as

$$F_i = f_i + \sum_{n=1}^{N} \lambda_{i,n} f_n \quad (9)$$

It is possible to produce null field on every node except at i^{th} node by using F_i as the adaptive node basis function. Hence, except the diagonal term, it produces whole i^{th} column of **[Z]-**matrix as zero, or near zero. For each node, considering it as a main node, the adaptive basis function need to be constructed, i.e. F_i, $i = 1, 2, ...N_n$ has to be constructed and to calculate matrix **[Z],** the corresponding elements of matrix **[Z]** should be determined**.** After turning all the elements that are less than threshold value to zero, the resulting Matrix **[Z]** is a diagonal matrix. Since the matrix **[Z]** is a diagonal matrix the $\mathbf{[Z]^{-1}}$ can be easily calculated.

The solution may or may not be accurate enough, once the solution **{X}** is obtained from $\mathbf{[Z]^{-1}} \times \mathbf{\{Y\}}$**.** The obtained solution may not be accurate if the number of nodes in the cluster is small. This is due to the reason that the off diagonal elements in the **[Z]** may be higher than the threshold value and hence may be a true diagonal matrix. Higher the number of nodes in the cluster, lower the value of the off diagonal elements in the matrix **[Z].** When the number of nodes in the cluster is small, there are two ways to get an accurate solution in those cases. They are:

I. Increase the number of nodes in the cluster, but this increases the computational time. It can be considered as a good number if the number of nodes in the cluster are higher than ~ 12% of total nodes on the surface of the object.

II. To produce accurate and a faster solution, use the solution obtained from the small number of nodes in the cluster as an initial guess to solve the independent node type basis function linear equation **[Z]{X} = {Y}.**

Features of Adaptive Node type basis functions:

The significant features of the adaptive node type basis functions are:

1. The [Z] matrix is only diagonal and elements are stored diagonally and hence the whole matrix need not be stored.
2. The computational time required to solve the **ZX = Y** is totally eliminated.
3. Only the N number of λ coefficients need to be stored for each node.
4. The nodes for the cluster need not be adjacent to the main node.
5. All the nodes for the cluster need not be physically present inside a boundary of the adaptive basis function.
6. The obtained solution may not be accurate incase number of nodes in the cluster is less than 12 %; but this can be used to solve the independent node type basis function solution as an initial guess. Still additionally **[Z]** matrix of the independent node type basis function solution need to be stored, but it is still worth storing additional data when benefit in the computational time is considered.
7. The time required to fill the **[Z]** matrix is an order less than that required to invert it. The operations required to fill a matrix for an object having N_n is N_n^2 and to invert, it requires N_n^3. With the usage of the adaptive node type basis functions, the number of operations required to invert the **[Z]** matrix is totally eliminated and hence N_n^3 is saved. Therefore the solution is faster than the independent node type basis function.
8. If the solution of the **[Z]** matrix that is based on the adaptive node type basis function is used as an initial

guess to solve the independent node type basis function, one may have to additionally solve the linear system of equations few times, which is still worth doing when compared with the benefit one gets since the inversion of **[Z]** matrix is totally eliminated.

IV NUMERICAL RESULTS

In this section, numerical results are presented for four cases of the spheres. The sphere of radius is modeled with 4 sizes of the mesh using triangular patch modeling. The temperature distributions are computed using the BEM solution with node basis functions and node based adaptive basis functions. Four different temperatures are specified on the surface of the sphere for four different cases.

Table 1: Model discretization using triangular patch modeling, Number of basis functions in the cluster and Number of iterations for each model to improve accuracy

Model Name	Nodes	Faces	Edges	Number of basis functions in the cluster				
sp10X10	92	180	270	5	7	11	15	19
sp12X12	134	264	396	5	11	17	21	27
sp15X15	212	420	630	9	17	25	35	43
sp20X20	382	760	1140	15	32	47	61	77
Number of iterations				20	16	12	8	4

Table 1 shows the different models used in this work for testing the solution. The models vary from just 92 nodes to 382 nodes on the surface. Since the basis functions are defined on the nodes, the sizes of the matrices are 92×92, 134×134, 212×212 and 382×382 respectively for the cases of sp10×10, sp12×12, sp15×15 and sp20×20. These are the sizes of the matrices that will result if the basis functions are defined only on one node. In this article, adaptive basis functions results are compared with results of single node basis functions.

Table 1 shows the number of basis functions defined in the cluster in various simulations. The number of basis functions have to be chosen to be the nearest odd number that is close to 4%, 8%, 12%, 16% and 20% of the total number of nodes in the surface. These simulations are run in order to assess the improvement in the solution when the number of basis functions are increased in the cluster.

The solution obtained using certain number of basis functions may or may not be accurate enough and will definitely have some deviation from the solution obtained using the single node basis functions. Hence this deviation needs to be bridged. The accuracy of the solution can be improved either by increasing the number of basis functions in the cluster or by using the solution obtained from the adaptive basis functions as the seed solution to the iterative solution of the single node basis function solution.

The number of iterations used to get faster and accurate solution is shown in Table 1. Higher the number of basis functions in the cluster, lower the number of iterations required to get the accurate solutions. Conversely, lower the number of basis functions used in the definition of the adaptive basis function, higher the deviation it will have from the accurate solution, and hence higher the number of iterations are required to improve the accuracy.

Fig 2 shows the distribution of temperature for a model sp10×10 up to distance of 10m from the surface of the sphere. It can be seen from the plot as the number of basis functions increase, the solution also improves its accuracy.

In Fig 2, single node basis functions are compared with adaptive basis functions results. 5basis_20itr model indicates, 5 basis functions are chosen into cluster to define the adaptive basis functions and 20 iterations are used to solve the single node basis function solution with the adaptive basis function solution. Similarly, other models also indicate the number of basis functions in the cluster and the number of iterations used.

By observing the graph, 11 basis functions and 12 iterations are the optimum combination of basis function iterations for sp10×10 model in the prediction of temperature on the surface of the sphere. The optimum combination at a distance of 1m and 2m is 19 basis functions with just 4 iterations.

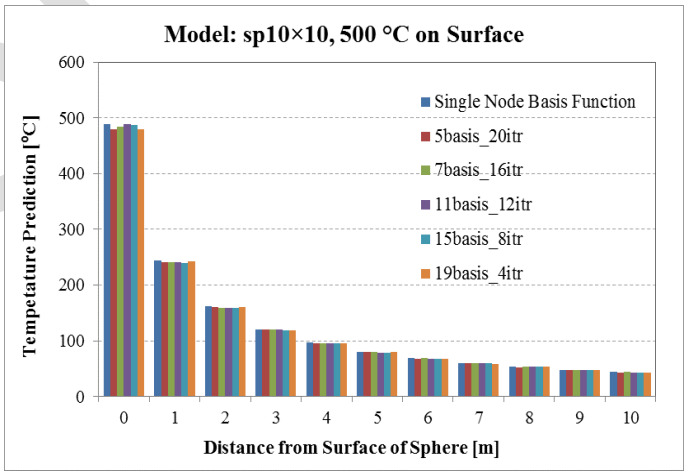

Figure 2: Temperature distributions at different distances from the surface of the sphere (model sp10×10)

In Fig 3, the error in the prediction of temperatures by adaptive basis functions with respect to single node basis functions are plotted. The results are plotted for a model of sp10×10. When a temperature of 500 °C is specified on the surface of the sphere, it can be seen from the Fig 6.3 that when 11 basis functions are used in the cluster, the error is very less for the temperature prediction on the surface of the sphere. The error is low at a distance of 1m and 2m from the surface of the sphere, when the 19 basis functions are used in the cluster.

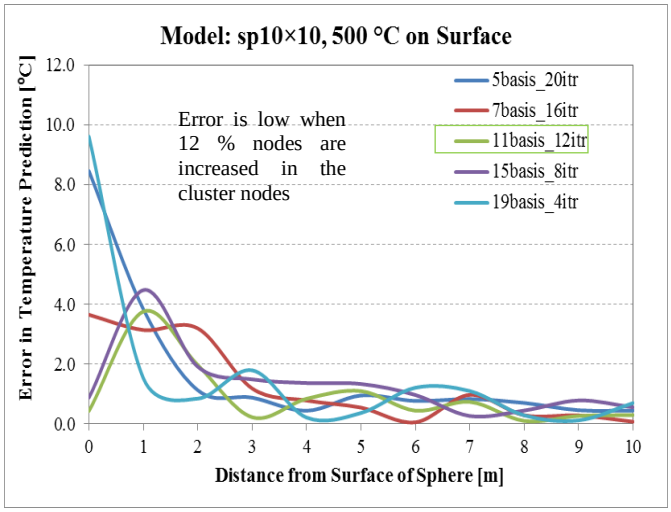

Figure 3: Distribution of error in temperature predictions at different distances from the surface of the sphere (model sp10×10)

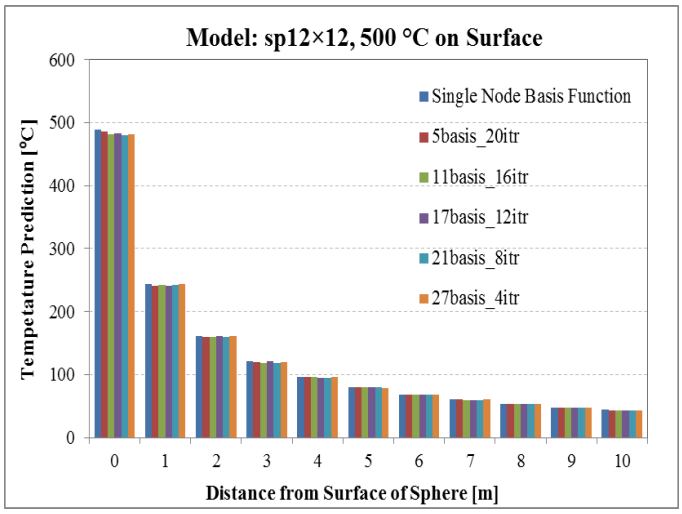

Figure 4: Temperature distributions at different distances from the surface of the sphere (model sp12×12)

Fig 4 shows the distribution of temperature for a model sp12×12 up to distance of 10m from the surface of the sphere.

By observing the Fig 4, the model with 5 basis functions and 4 iterations are the optimum combination of basis function-iterations that are predicted at temperature of 486 °C on the surface of the sphere for the sp12×12 model. The optimum combination at distances of 1m and 2m are 27 basis functions with 4 iterations and 17 basis functions with 12 iterations respectively.

Figure 4 shows temperature distribution for a temperature, defined on the surface of the sphere when the sphere is approximated with triangular mesh with model sp12×12, up to a distance of 10m from the surface of the sphere. There is good agreement of the prediction of temperatures using the adaptive basis functions when compared with the single node solution. There is slight drop in the temperature by 2.4 °C on the surface of the temperature when 5 basis functions are used in the cluster. At a distance of 1m from the surface of the sphere, the drop is 3.7 °C, but the drop decreases thereafter.

In Fig. 5, the errors are plotted for the temperature predictions by adaptive basis functions with respect to the results obtained using single node basis functions. Again, the results are shown for a model of sp12×12 for a temperature definition of 500 °C on the surface of the sphere. It can be observed from the Fig. 5 that on the surface of the sphere, the maximum error is produced by 21 basis functions model, and 27 basis functions model the next highest. The lowest error is produced by 5 basis functions model and 17 basis functions model. At a distance of 1m from the surface of the sphere, the 5 basis functions model and 17 basis functions model produces highest error. As the distance from the surface increases, the error falls to less than 1 °C.

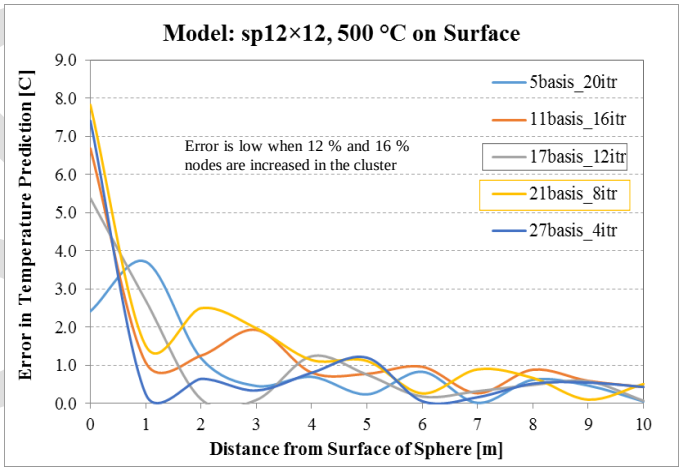

Figure 5: Distribution of error in temperature predictions at different distances from the surface of the sphere (model sp12×12)

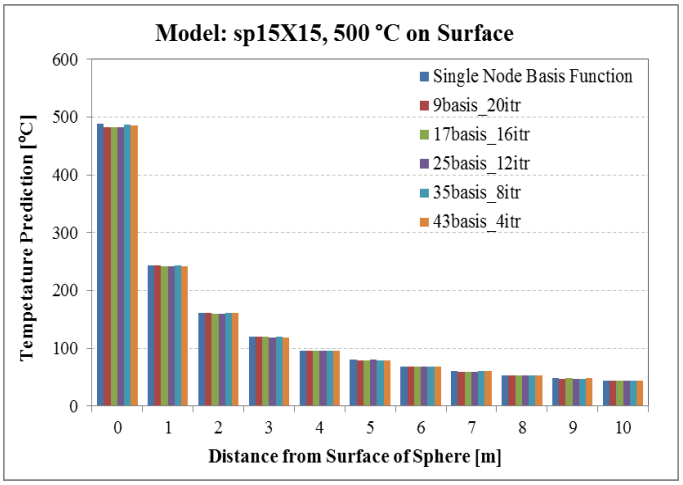

Figure 6: Distribution of error in temperature predictions at different distances from the surface of the sphere (model sp15×15)

Fig 6 shows the distribution of temperature for a model sp15×15 up to distance of 10m from the surface of the sphere.

In Fig 6, it is observed that at a distance of 1m and 2m from the surface of the sphere, the optimum model that produced the closest result with respect to the single node basis function solution is 25basis_12itr. On the surface of the sphere and at a distance of 3m from the surface of the sphere, model 35basis_8itr has given best results.

Fig 7 shows the error in the prediction of temperature at a distance up to 10m from the surface of the sphere for sp15×15 model. The model 25basis_12itr has performed best at 1m and 2m from the surface of the sphere and 35basis_8itr performed best at 0m and 3m from the surface. Overall, the 25basis_12itr and 35basis_8itr has performed well at many locations than the other models.

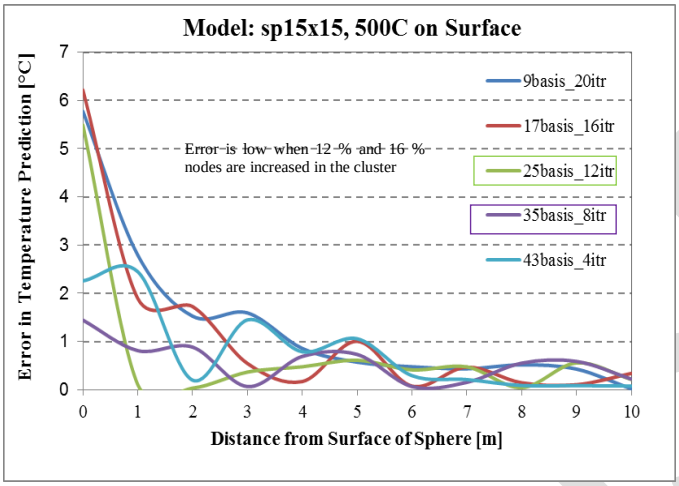

Figure 7: Distribution of error in temperature predictions at different distances from the surface of the sphere (model sp15×15)

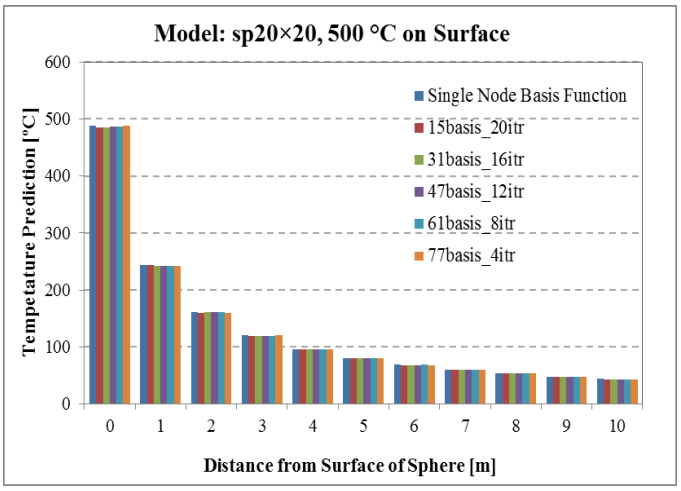

Figure 8: Distribution of error in temperature predictions at different distances from the surface of the sphere (model sp20×20)

Figs. 8 and 9 show the distributions of temperature and the associated errors in the predictions respectively for the model sp20×20 when the temperature of 500 °C is defined on the surface of the sphere. The lowest error is produced by 47basis_12itr basis functions model and has performed well at many locations than the other models.

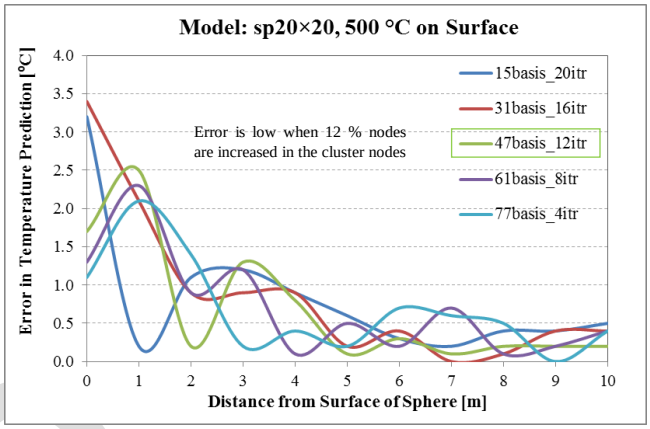

Figure 9: Distribution of error in temperature predictions at different distances from the surface of the sphere (model sp20×20)

V. CONCLUSION

In this work, Adaptive basis function is defined on the standard BEM solution procedure in contrast to using the regular basis functions used in FEM/BEM procedures. The advantages of following such procedure are explained. With the adaptive basis function, only few iterations are required to solve the problem and hence it is much faster than solving a full matrix. It is proven that the results predicted by the adaptive basis functions match well with that of the single node basis functions. In this article, effort is made to improve the accuracy of the solution by increasing number of nodes in the cluster. Different combinations are analyzed by changing the number of nodes and iterations. Result will be accurate if the number of nodes in the cluster are higher than 12 % of total number of nodes on the surface of the object.

Model	Combination		
	Optimum-1	Optimum-2	Recommended
sp10×10	11basis_12itr	19basis_8itr	11basis_12itr
sp12×12	5basis_20itr	17basis_12itr	17basis_12itr
sp15×15	25basis_12itr	35basis_8itr	25basis_12itr
sp20×20	47basis_12itr	77basis_4itr	47basis_12itr

***Recommended no. of nodes is 12 % higher than total nodes on the surface of the object**

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