A Note on M_2 *Partitions* of n

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Abstract: **S.Ahlgren, Bringmann and Lovejoy [1] defined M_2 *spt(n)* to be the number of smallest parts in the partitions of n without repeated odd parts and with smallest part even and Bringmann, Lovejoy and Osburn [4]derivedthe generating function for M_2 *spt(n)* .Hanumareddy and Manjusri [7] derived generating function for the number of smallest parts of Partitions of *n* by using *r- Partitions* of *n*. Hanumareddy and Gudimella V R K Sagar[9], [10] defined M_2 *partitions* and *Ga partitions* of positive integer n. In this chapter we derived generating function for M_2 *spt(n)* by using *r-* M_2 *partitions* of n. We also derive generating function for sum M_2 *spt(n)*.**

Keywords: **Partition, *r- Partition*, M_2 *partitions*, Smallest part of the M_2 *partitions*.**

Subject classification: **11P81 Elementary theory of Partitions.**

I. INTRODUCTION

Let $M2\xi(n)$ be denote the set of all *partitions* of n with odd numbers appears at most one time and smallest parts are even numbers. Let $M2p(n)$ be the cardinality of $M2\xi(n)$ for $n\in N$ and $M2p(0)=1$. If $1\le r\le n$, write $M2p_r(n)$ for the number of *partitions* of n in $M2\xi(n)$ each consisting of exactly r parts, i.e $r-M2\,partitions$ of n in $M2\xi(n)$. If $r\le 0$ or $r\ge n$, we write $M2p_r(n)=0$. Let $M2p(k,n)$ represent the number of *partitions* of n in $M2\xi(n)$ using natural numbers atleast as large as k only. Let the *partitions* in $M2\xi(n)$ be denoted by $M2\,partitions.$

Let $M2spt(n)$ be denotes the number of smallest parts including repetitions in all *partitions* of n in $M2\xi(n)$ and $sumM2spt(n)$ be denotes the sum of the smallest parts. For $i\ge 1$,let us adopt the following notation on the lines of [3].

$M2m_s(\lambda)=$number of smallest parts of λ in $M2\xi(n)$.

$$M2spt(n)=\sum_{\lambda\in\xi(n)} M2m_s(\lambda)$$

For example $M2\xi(10)$:

$M2p(10)=8$ $\qquad$ $M2spt(10)=15$

$\underline{10}$, $8+\underline{2}$, $6+\underline{4}$, $6+\underline{2}+\underline{2}$, $5+3+\underline{2}$, $4+4+\underline{2}$, $4+\underline{2}+\underline{2}+\underline{2}$, $\underline{2}+\underline{2}+\underline{2}+\underline{2}+\underline{2}$.

Let $M\xi(n)$ be denote the set of all *partitions* of n with odd numbers appears at most one time and part 1's not appear and $Mp(n)$ the cardinality of $M\xi(n)$. Let $Mp_r(n)$ for the number of $r-partitions$ of n in $M\xi(n)$. Let $Mp(k,n)$ represent the number of *partitions* of n in $M\xi(n)$ using natural numbers atleast as large as k only. Let $Mspt(n)$ denote the number of smallest parts including repetitions in all *partitions* of n in $M\xi(n)$. Let $Mm_s(\lambda)$ be number of smallest parts of λ in $M\xi(n)$ and $Mspt(n)$ is $\sum_{\lambda\in\xi(n)} Mm_s(\lambda)$

For example $M\xi(9)$:

$Mp(9)=7$ $\qquad$ $M2spt(9)=8$

9, $7+\underline{2}$, $6+3$, $5+\underline{4}$, $5+2+\underline{2}$, $4+3+\underline{2}$, $3+\underline{2}+\underline{2}+\underline{2}$.

We observe that

1.1 The generating function for the number of $r-M2\,partitions$ of n with odd numbers appears at most one time and smallest parts are even numbers is

$$M2p_r(n)=\frac{q^{2r}\left(-q,q^2\right)_{r-1}}{\left(q^2,q^2\right)_r} \tag{1.2}$$

1.2 The generating function for the number of $r-partitions$ of n with odd numbers appears at most one time and 1's not appear is

$$Mp_r(n) = \frac{q^{2r}\left(-q, q^2\right)_r}{\left(q^2, q^2\right)_r} \tag{1.3}$$

II. GENERATING FUNCTION FOR $M2spt(n)$

The generating function for the number of smallest parts of all partitions of positive integer *n* is derived by G.E.Andrews. By utilizing $r - M2\,partitions$ of *n*, we propose a formula for finding the number of smallest parts of $M2\,partitions$ of *n*.

2.1Theorem:

$$M2spt(n) = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} Mp(2k, n-2tk) + d\left(\frac{n}{2}\right)$$

where $d(n)$ is the number of positive divisors of n.

Proof:[5] Let $n = (\lambda_1, \lambda_2, \dots, \lambda_r) = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}, k_1^{\alpha_l}\right)$, $\left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}, k_1^{\alpha_l}\right) \in M2\xi(n), k_1 = 2k, k \in N$ be any $r - M2\,partition$ of n with *l* distinct parts such that odd parts not repeated and smallest parts are even numbers.

Case 1:[6] Let $r > \alpha_l = t$ which implies $\lambda_{r-t} > k_1$. Subtract all $k_1's$, we get $n - tk_1 = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}\right)$, $\left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}\right) \in M\xi(n)$

Hence $n - tk_1 = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t) - partition$ of $n - tk_1$ with l - 1 distinct parts and each part is greater than or equal to $k_1 + 1$. Here the number of $r - M2\,partitions$ with smallest part k_1 that occurs exactly *t* times among all $r - M2\,partitions$ of *n* is $Mp_{r-t}(k_1 + 1, n - tk_1)$.

Case 2: Let $r > \alpha_l > t$ which implies $\lambda_{r-t} = k_1$

Omit $k_1's$ from last t places, we get $n - tk_1 = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}, k_1^{\alpha_l - t}\right)$, $\left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}, k_1^{\alpha_l - t}\right) \in M2\xi(n)$. Hence $n - tk_1 = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}, k_1^{\alpha_l - t}\right)$ is a $(r-t) - M2\,partition$ of $n - tk_1$ with l distinct parts and the least part is k_1.

Now we get the number of $r - M2\,partitions$ with smallest part k_1 that occurs more than *t* times among all $r - M2\,partitions$ of *n* is $M2\,f_{r-t}(k_1, n - tk_1)$.

Case 3: Let $r = \alpha_l = t$ which implies all parts in the *partition* are equal.

The number of *partitions* of *n*with equal parts in set $\{M2\xi(n), k_1 \in 2N\}$ is equal to the number of divisors of $\frac{n}{2}$. Since the number of divisors of $\frac{n}{2}$ is $d\left(\frac{n}{2}\right)$, the number of $partitions$ of *n* with equal parts in set $\{M2\xi(n), k_1 \in 2N\}$ is $d\left(\frac{n}{2}\right)$, .

$$\text{where} \quad \beta = \begin{cases} 1 & \text{if } 2r \mid n \\ 0 & \text{otherwise} \end{cases}$$

From cases(1), (2) and (3) we get $r - partitions$ of *n* with smallest part k_1 that occurs *t* times is

$$Mp_{r-t}(k_1 + 1, n - tk_1) + M2\,f_{r-t}(k_1, n - tk_1) + \beta$$

$$= Mp_{r-t}(k_1, n - tk_1) + \beta$$

The number of smallest parts in $M2\,partitions$ of *n* is

$$M2spt(n) = \sum_{k_1=1}^{\infty}\sum_{t=1}^{\infty} Mp(k_1, n - tk_1) + d\left(\frac{n}{2}\right)$$

$$\Rightarrow M2spt(n) = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} Mp(2k, n - 2tk) + d\left(\frac{n}{2}\right)$$

2.2 Theorem: $Mp_r(2k+2, n) = Mp_r(n - 2kr)$

Proof: Let $n = (\lambda_1, \lambda_2, \dots, \lambda_r), \lambda_i > k_1 + 1 \quad \forall i$, $k_1 = 2k, k \in N$ be any $r - M2\,partition$ of n such that odd numbers not repeated and smallest parts are even numbers. Subtracting k_1 from each part, we get

$$n-k_1r=\left(\lambda_1-k_1,\lambda_2-k_1,\ldots,\lambda_r-k_1\right)$$

Hence $n-k_1r=\left(\lambda_1-k_1,\lambda_2-k_1,\ldots,\lambda_r-k_1\right)$ is a $r-M2\,partition$ of $n-k_1r$ with odd parts not repeated and 1's not appear.

Therefore the number of $r-M2\,partitions$ of n with parts greater than or equal to k_1+2 is $Mp_r\left(n-k_1r\right)$.

Hence $Mp_r\left(k_1+2,n\right)=Mp_r\left(n-k_1r\right)$

$$\Rightarrow Mp_r\left(2k+2,n\right)=Mp_r\left(n-2kr\right)$$

2.3 Theorem:

$$\sum_{n=1}^{\infty}M2spt\left(n\right)q^n=\sum_{n=1}^{\infty}\frac{q^{2n}\left(-q^{2n+1};q^2\right)_\infty}{\left(1-q^{2n}\right)^2\left(q^{2n+2};q^2\right)_\infty}$$

Proof: From theorem (2.1) we have

$$M2spt\left(n\right)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}Mp\left(2k,n-2tk\right)+d\left(\frac{n}{2}\right)$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}Mp_r\left(2k,n-2tk\right)+d\left(\frac{n}{2}\right)$$

first replace $2k+2$ by $2k$, then replace n by $n-2tk$ in theorem (2.2)

$$=\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{k=1}^{\infty}Mp_r\left(n-2tk-r\left(2k-2\right)\right)+d\left(\frac{n}{2}\right)$$

Where $d\left(n\right)$ is the number of positive divisors of n.

From (1.5.1) in [8]

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{2r+2tk+r(2k-2)}\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}+\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{2tk+2kr}\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}+\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{2tk}\left[\sum_{r=1}^{\infty}\frac{\left(q^{2k}\right)^r\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}\right]+\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}\left[\sum_{r=1}^{\infty}\frac{\left(q^{2k}\right)^r\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}\right]+\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}\left[1+\sum_{r=1}^{\infty}\frac{\left(q^{2k}\right)^r\left(-q,q^2\right)_r}{\left(q^2,q^2\right)_r}\right]$$

Put $t=q^{2k}, a=-q, q=q^2$

in theorem 2.1 'The Theory of partitions' by G.E.Andrews

$$=\sum_{k=1}^{\infty}\frac{q^{2k}}{\left(1-q^{2k}\right)}\prod_{r=0}^{\infty}\frac{\left(1+q^{2r+2k+1}\right)}{\left(1-q^{2r+2k}\right)}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k}\left(1+q^{2k+1}\right)\left(1+q^{2k+3}\right)\left(1+q^{2k+5}\right)\ldots}{\left(1-q^{2k}\right)\left(1-q^{2k}\right)\left(1-q^{2k+2}\right)\left(1-q^{2k+4}\right)\left(1-q^{2k+6}\right)\ldots}$$

$$=\sum_{k=1}^{\infty}\frac{q^{2k}\left(-q^{2k+1};q^2\right)_\infty}{\left(1-q^{2k}\right)^2\left(q^{2k+2};q^2\right)_\infty}$$

$$\sum_{n=1}^{\infty}M2spt\left(n\right)q^n=\sum_{n=1}^{\infty}\frac{q^{2n}\left(-q^{2n+1};q^2\right)_\infty}{\left(1-q^{2n}\right)^2\left(q^{2n+2};q^2\right)_\infty}$$

2.4 Corollary: The generating function for $cM2spt\left(n\right)$, the number of smallest parts of the $M2\,partitions$ of n which are multiples of c is

$$\sum_{n=1}^{\infty}cM2spt\left(n\right)q^n=\sum_{n=1}^{\infty}\frac{q^{2cn}\left(-q^{2cn+1};q^2\right)_\infty}{\left(1-q^{2cn}\right)^2\left(q^{2cn+2};q^2\right)_\infty}$$

2.5 Theorem: The generating function for the sum of smallest parts of the $M2\,partitions$ of n is

$$\sum_{n=1}^{\infty}sumM2spt\left(n\right)q^n=\sum_{n=1}^{\infty}\frac{2nq^{2n}\left(-q^{2n+1};q^2\right)_\infty}{\left(1-q^{2n}\right)^2\left(q^{2n+2};q^2\right)_\infty}$$

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