Generating Function for $\overline{Ga\,spt(n)}^{j}$

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Abstract: In this section, we derive a formula for the generating function the number of smallest parts including repetitions in all j^{th} over Ga partition of n.

I. INTRODUCTION

Corteal and Love joy [1] initiated the study of over partitions. Hanuma Reddy and Janakamma [3] derived a formula for the number of i^{th} over Ga partition of n

When the parts are in A.P. In this paper, we obtain a formula for the generating function for the number of smallest parts including repetitions in all j^{th} over Ga partition of n.

1.1 Definitions and notation:

1) A r-partition of n is a non-deceasing sequence of positive integers whose sum is =n. Each number is called partition
2) A j^{th} over *r – partition of n is r-partition of n in which a part is over lined j times at its first* appearances.
3) The cardinality of the set of j^{th} over *r – partition*s of *n is* denoted by $\overline{p_r(n)}^{j}$.
4) $\overline{p_r(s,n)}^{j}$: The number of j^{th} over *partitions* of n with least part greater than or equal to s is defined by $\overline{p_r(s,n)}^{j}$.
5) Ga partition : A partition of *n* is a *Ga partition* if smallest parts are of the form $a^{k-1}, k \in N$.
6) over Ga partition Is a Ga partition in which first (equivalently, the final) occurrence of a part is over lined up to j times successively.
7) $\overline{Ga\,spt(n)}^{j}$: denotes the number of smallest parts including repetitions in all j^{th} Over *r – partition* of *n*
8) $(a,q)_\infty$: $\prod_{n=1}^{\infty}\left(1-aq^{n}\right)$
9) $(q)_\infty$: $(1,q)_\infty$
10) $d(a,n)$ is the number of divisors of n of the form? $a^{k-1}, k \in N$

II. GENARATING FUNCTIONS:

In this section, we obtain a formula for the generating function for the number of smallest parts including repetitions in all j^{th} over Ga partitions of n

2.1 Proposition: The number of partitions of n *into* r *parts each* $> a^{k-1}$ *= number of partitions of* $n - a^{k-1}r$ *into* r *parts.*

$$i.e\ \overline{p_r\left(a^{k-1}+1, n\right)}^{j} = \overline{p_r\left(n - a^{k-1}r\right)}^{j} \qquad (2.3(i))$$

Proof: Let $n = (\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_r > a^{k-1}$, write

$n - ra^{k-1} = \left(\mu_1, \mu_2, \ldots\ldots \mu_r\right)$ where $\mu_i = \lambda_i - a^{k-1}$.Futher if $n - ra^{k-1} = \left(\mu_1, \mu_2, \ldots\ldots \mu_r\right)$ *and* $\lambda_i = \mu_i + a^{k-1}$ *then* $n = \left(\lambda_1, \ldots\ldots \lambda_r\right)$ cleary the correspondence $\left(\lambda_1, \ldots\ldots \lambda_r\right) \leftrightarrow \left(\mu_1, \ldots\ldots \mu_r\right)$

is one one and onto from $r - partitions\ of\ n$ with smallest part greater than a^{k-1} and all $r - partitions\ of\ n - a^{k-1}$.this one- one correspondence yields the required equality.

2.2 Corollary: For $j = 1$ the number of r-*over partitions* of *n* with parts, greater than or equal to $a^{k-1}+1$ is the number of $r - overpartitions$ of $n - a^{k-1}r$ parts.

We now derive the generating function for the number of smallest parts of all $j^{th}\ overGa\ partitions$ of *n* with the help of $r - j^{th}\ overGa\ partitions$ of n.

2.3 Theorem: **T**he generating function for $\overline{Ga\,spt(n)}^{j}$ is

$$\sum_{n=1}^{\infty} \overline{Ga\,spt(n)}^{j} q^n = \frac{(-j,q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(j+1)q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)} \frac{(q)_{a^{n-1}-1}}{(-j,q)_{a^{n-1}+1}} \ldots\ldots\ldots\ldots(A)$$

Proof: From theorem (3.1) of [3] we have

$$\overline{Ga\,spt(n)}^{j} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} \overline{p\left(a^{k-1}, n - ta^{k-1}\right)}^{j} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty} \overline{p\left(a^{k-1}+1, n - ta^{k-1}\right)}^{j} + (j+1)d(a,n)$$

Replace $a^{k-1}+1$ by a^{k-1}, n by $n - ta^{k-1}$ for first part and n by $n - ta^{k-1}$ for second part in (2.5) From

$$\overline{Ga\,spt(n)}^{j} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} \overline{p_r\left(n - ta^{k-1} - r\left(a^{k-1}-1\right)\right)}^{j} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} \overline{p_r\left(n - ta^{k-1}k - ra^{k-1}\right)}^{j} + (j+1)d(a,n)$$

$$\sum_{n=1}^{\infty} \overline{Ga\,spt(n)q^n}^{j}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} \frac{q^{r+ta^{k-1}+r\left(a^{k-1}-1\right)}(-j,q)_r}{(q)_r} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} \frac{q^{r+ta^{k-1}+ra^{k-1}}(-j,q)_r}{(q)_r} + \sum_{k=1}^{\infty} \frac{(j+1)q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} \frac{q^{ta^{k-1}+ra^{k-1}}(-j,q)_r}{(q)_r} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty} \frac{q^{r+ta^{k-1}+ra^{k-1}}(-j,q)_r}{(q)_r} + \sum_{k=1}^{\infty} \frac{(j+1)q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{ta^{k-1}}\left[\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^{r}(-j,q)_{r}}{(q)_{r}}\right]+j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{ta^{k-1}}\left[\sum_{r=1}^{\infty}\frac{q^{r}\left(q^{a^{k-1}}\right)^{r}(-j,q)_{r}}{(q)_{r}}\right]+\sum_{k=1}^{\infty}\frac{(j+1)q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left[\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^{r}(-j,q)_{r}}{(q)_{r}}\right)-1\right]$$

$$+j\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left[\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}+1}\right)^{r}(-j,q)_{r}}{(q)_{r}}\right)-1\right]+\sum_{r=1}^{\infty}\frac{(j+1)q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^{r}(-j,q)_{r}}{(q)_{r}}\right)+j\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}+1}\right)^{r}(-j,q)_{r}}{(q)_{r}}\right)$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r}q^{a^{k-1}}}{1-q^{r}q^{a^{k-1}}}\right)+j\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r}q^{a^{k-1}+1}}{1-q^{r}q^{a^{k-1}+1}}\right)\qquad \text{from [3]}$$

$$=\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r+a^{k-1}}}{1-q^{r+a^{k-1}}}\right)+j\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r+a^{k-1}+1}}{1-q^{r+a^{k-1}+1}}\right)$$

$$=\frac{(-j,q)_{\infty}}{(q)_{\infty}}\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}-1}}{(-j,q)_{a^{k-1}}}+j\frac{(-j,q)_{\infty}}{(q)_{\infty}}\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}}}{(-j,q)_{a^{k-1}+1}}$$

$$=\frac{(-j,q)_{\infty}}{(q)_{\infty}}\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}-1}}{(-j,q)_{a^{k-1}}}\left[1+j\frac{\left(1-q^{a^{k-1}}\right)}{\left(1+jq^{a^{k-1}}\right)}\right]$$

$$=\frac{(-j,q)_{\infty}}{(q)_{\infty}}\sum_{k=1}^{\infty}\frac{(j+1)q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}-1}}{(-j,q)_{a^{k-1}+1}}$$

$$=\frac{(-j,q)_{\infty}}{(q)_{\infty}}\sum_{n=1}^{\infty}\frac{(j+1)q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)}\frac{(q)_{a^{n-1}-1}}{(-j,q)_{a^{n-1}+1}}$$

2.3.4 Illustration: We explain our theorem by an illustration. In this context we often come across with terms of the form $\frac{1}{1\pm x}$. we replace these terms by the power series expansions

$$\frac{1}{1\pm x}=1\pm x\pm x^2\pm\ldots\pm x^n\pm\ldots$$

(With alternate $-and+$ signs for $1+x$ and $+$ signs for $1-x$ in the denominator.)

The coefficient of x^n in this context can be found with the help of Math lab.

When j = 2 and a = 3 in theorem 2.3.3 we have

$$\sum_{n=1}^{\infty}\overline{G3spt(n)}^{2}q^n=\frac{(-2,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\left[\frac{(2+1)q^{3^{n-1}}}{\left(1-q^{3^{n-1}}\right)}\cdot\frac{(q)_{3^{n-1}-1}}{(-2,q)_{3^{n-1}+1}}\right]$$

$$=\frac{(-2,q)_\infty}{(q)_\infty}\left[\frac{(2+1)q}{(1-q)(1+2)(1+2q)}+\frac{(2+1)q^3(1-q)(1-q^2)}{(1-q^3)(2+1)(1+2q)(1+2q^2)(1+2q^3)}\right.$$

$$\left.+\frac{(2+1)q^9(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^7)(1-q^8)}{(1-q^9)(2+1)(1+2q)(1+2q^2)(1+2q^3)(1+2q^4)(1+2q^5)(1+2q^6)(1+2q^7)(1+2q^8)(1+2q^9)}+\ldots\right.$$

$$=\frac{(-2,q)_\infty}{(q)_\infty}\left[\frac{q}{(1-q)(1+2q)}+\frac{q^3(1-q)(1-q^2)}{(1-q^3)(1+2q)(1+2q^2)(1+2q^3)}\right.$$

$$\left.+\frac{q^9(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^7)(1-q^8)}{(1-q^9)(1+2q)(1+2q^2)(1+2q^3)(1+2q^4)(1+2q^5)(1+2q^6)(1+2q^7)(1+2q^8)(1+2q^9)}+\ldots\right.$$

$$=\frac{(-2,q)_\infty}{(q)_\infty}\left[q-q^2+4q^3-8q^4+14q^5-25q^6+58q^7-118q^8+226q^9-448q^{10}+\ldots\right]$$

$$=3\left(1+3q+6q^2+15q^3+27q^4+51q^5+93q^6+\ldots\right)$$

$$\left(q-q^2+4q^3-8q^4+14q^5-25q^6+58q^7-\ldots\right)$$

$$=3\left(q+2q^2+7q^3+13q^4+26q^5+53q^6+97q^7+\ldots\right)$$

$$=3q+6q^2+21q^3+39q^4+78q^5+159q^6+291q^7+\ldots$$

*2.3.5 Corollary***:** The generating function for $\overline{Ga\,A_c(n)}^{j}$, the number of smallest parts of the j^{th}*overGa partitions* of n which are multiples of c is

$$\sum_{n=1}^{\infty}\overline{Ga\,A_c(n)q^n}^{j}=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(j+1)q^{ca^{n-1}}}{\left(1-q^{ca^{n-1}}\right)}\frac{(q)_{ca^{n-1}-1}}{(-j,q)_{ca^{n-1}+1}}$$

2.3.6 Corollary: The generating function for $\overline{Ga\,A_c(n)}$, the number of smallest parts of the *overGa partitions* of n which are multiples of c is

$$\sum_{n=1}^{\infty}\overline{Ga\,A_c(n)q^n} = \frac{(-1,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{2q^{ca^{n-1}}}{\left(1-q^{ca^{n-1}}\right)}\frac{(q)_{ca^{n-1}-1}}{(-1,q)_{ca^{n-1}+1}}$$

To evaluate the sum of smallest parts of $r-j^{th}overGa\ partitions$ of n by applying the concept of $r-j^{th}overGa\ partitions$ of *n*, we propose the following theorem.

*2.3.7 Theorem***:** The generating function for the sum of smallest parts of $j^{th}\ overGa\ partitions$ of n is

$$\sum_{n=1}^{\infty}\overline{sum\,Ga\,spt(n)q^n}^{\,j} = \frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=t_1}^{\infty}\frac{(j+1)a^{n-1}q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)}\frac{(q)_{a^{n-1}-1}}{(-j,q)_{a^{n-1}+1}}$$

Proof: **T**he sum of smallest parts of $j^{th}\ overGa\ partitions$ of a positive integer n is

$$\overline{sum\ \ Ga\,spt(n)}^{\,j} = \sum_{t=1}^{\infty}\sum_{k=1}^{\infty}a^{k-1}\ \overline{p\left(a^{k-1},n-ta^{k-1}\right)}^{\,j} + j\sum_{t=1}^{\infty}\sum_{k=1}^{\infty}a^{k-1}\ \overline{p\left(a^{k-1}+1,n-ta^{k-1}\right)}^{\,j}$$
$$+(j+1)d(a,n)$$
$$= \sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{\infty}a^{k-1}\ \overline{p_r\left(a^{k-1},n-ta^{k-1}\right)}^{\,j} + j\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{\infty}a^{k-1}\ \overline{p_r\left(a^{k-1}+1,n-ta^{k-1}\right)}^{\,j}$$
$$+(j+1)d(a,n)$$

First replace $a^{k-1}+1$ by a^{k-1}, then replace n by $n-ta^{k-1}$ in (2.1.1)

$$= \sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{k=t_1}^{\infty}\overline{a^{k-1}.p_r\left(n-ta^{k-1}-r\left(a^{k-1}-1\right)\right)}^{\,j}$$
$$+ j\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{k=t_1}^{\infty}\overline{a^{k-1}.p_r\left(n-ta^{k-1}-ra^{k-1}\right)}^{\,j}$$
$$+(j+1)d(a,n)$$

Hence From (2.2(i)) the generating function for the sum of smallest parts of the $r-j^{th}overGa\ partitions$ of n.

$$= \sum_{k=1}^{\infty}\frac{a^{k-1}q^{a^{k-1}}}{1-q^{a^{k-1}}}\left[\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^r(-j,q)_r}{(q)_r}\right]$$
$$+ j\sum_{k=1}^{\infty}\frac{a^{k-1}q^{a^{k-1}}}{1-q^{a^{k-1}}}\left[\sum_{r=1}^{\infty}\frac{q^r\left(q^{1+a^{k-1}}\right)^r(-j,q)_r}{(q)_r}\right] + \sum_{k=1}^{\infty}\frac{(j+1)a^{k-1}q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$= \sum_{k=1}^{\infty}\frac{a^{k-1}q^{a^{k-1}}}{1-q^{a^{k-1}}}\left[1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^r(-j,q)_r}{(q)_r}\right]$$
$$+ j\sum_{k=1}^{\infty}\frac{a^{k-1}q^{a^{k-1}}}{1-q^{a^{k-1}}}\left[1+\sum_{r=1}^{\infty}\frac{q^r\left(q^{1+a^{k-1}}\right)^r(-j,q)_r}{(q)_r}\right]$$

$$=\sum_{k=1}^{\infty}\frac{a^{k-1}q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r}q^{a^{k-1}}}{1-q^{r}q^{a^{k-1}}}\right)+j\sum_{k=1}^{\infty}\frac{a^{k-1}q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r}q^{a^{k-1}+1}}{1-q^{r}q^{a^{k-1}+1}}\right)\qquad (\text{from}[3])$$

$$=\frac{(-j,q)_{\infty}}{(q)_{\infty}}\sum_{n=1}^{\infty}\frac{(j+1)a^{n-1}q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)}\frac{(q)_{a^{n-1}-1}}{(-j,q)_{a^{n-1}+1}}$$

$$\sum_{n=1}^{\infty}\overline{sum\,Ga\,spt(n)q^{n}}^{\,j}=\frac{(-j,q)_{\infty}}{(q)_{\infty}}\sum_{n=1}^{\infty}\frac{(j+1)a^{n-1}q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)}\frac{(q)_{a^{n-1}-1}}{(-j,q)_{a^{n-1}+1}}$$

*2.8 Illustration***:** When $j = 2$, $a = 2$ by 2.7 we have

$$\sum_{n=1}^{\infty}\overline{sum\,G2\,spt(n)q^{n}}^{\,j}=\frac{(-2,q)_{\infty}}{(q)_{\infty}}\sum_{n=1}^{\infty}\frac{(2+1)2^{n-1}q^{2^{n-1}}}{\left(1-q^{2^{n-1}}\right)}\frac{(q)_{2^{n-1}-1}}{(-j,q)_{2^{n-1}+1}}$$

By using math lab the R.H.S can be simplified into

$$3q+12q^{2}+18q^{3}+63q^{4}+96q^{5}+189q^{6}+336q^{7}+645q^{8}+1053q^{9}+1725q^{10}+\dots$$

The sum of smallest parts of second *overG2 partitions* of 6 is 225 and is verified from the following

$$5+\underline{1},\quad \bar{5}+\underline{1},\quad \bar{\bar{5}}+\underline{1},\quad 5+\underline{\bar{1}},\quad \bar{5}+\underline{\bar{1}},\quad \bar{\bar{5}}+\underline{\bar{1}},$$

$$5+\underline{\bar{\bar{1}}},\quad \bar{5}+\underline{\bar{\bar{1}}},\quad \bar{\bar{5}}+\underline{\bar{\bar{1}}},\quad 4+\underline{2},\quad \bar{4}+\underline{2},\quad \bar{\bar{4}}+\underline{2},$$

$$4+\underline{\bar{2}},\quad \bar{4}+\underline{\bar{2}},\quad \bar{\bar{4}}+\underline{\bar{2}},\quad 4+\underline{\bar{\bar{2}}},\quad \bar{4}+\underline{\bar{\bar{2}}},\quad \bar{\bar{4}}+\underline{\bar{\bar{2}}},$$

$$4+\underline{1}+\underline{1},\quad \bar{4}+\underline{1}+\underline{1},\quad \bar{\bar{4}}+\underline{1}+\underline{1},\quad 4+\underline{\bar{1}}+\underline{1},\quad \bar{4}+\underline{\bar{1}}+\underline{1},\quad \bar{\bar{4}}+\underline{\bar{1}}+\underline{1},$$

$$4+\underline{\bar{\bar{1}}}+\underline{1},\quad \bar{4}+\underline{\bar{\bar{1}}}+\underline{1},\quad \bar{\bar{4}}+\underline{\bar{\bar{1}}}+\underline{1},\quad 3+2+\underline{1},\quad \bar{3}+2+\underline{1},\quad \bar{\bar{3}}+2+\underline{1},$$

$$3+\bar{2}+\underline{1},\quad \bar{3}+\bar{2}+\underline{1},\quad \bar{\bar{3}}+\bar{2}+\underline{1},\quad 3+\bar{\bar{2}}+\underline{1},\quad \bar{3}+\bar{\bar{2}}+\underline{1},\quad \bar{\bar{3}}+\bar{\bar{2}}+\underline{1},$$

$$3+2+\underline{\bar{1}},\quad \bar{3}+2+\underline{\bar{1}},\quad \bar{\bar{3}}+2+\underline{\bar{1}},\quad 3+\bar{2}+\underline{\bar{1}},\quad \bar{3}+\bar{2}+\underline{\bar{1}},\quad \bar{\bar{3}}+\bar{2}+\underline{\bar{1}},$$

$$3+\bar{\bar{2}}+\underline{\bar{1}},\quad \bar{3}+\bar{\bar{2}}+\underline{\bar{1}},\quad \bar{\bar{3}}+\bar{\bar{2}}+\underline{\bar{1}},\quad 3+2+\underline{\bar{\bar{1}}},\quad \bar{3}+2+\underline{\bar{\bar{1}}},\quad \bar{\bar{3}}+2+\underline{\bar{\bar{1}}},$$

$$3+\bar{2}+\underline{\bar{\bar{1}}}\quad \bar{3}+\bar{2}+\underline{\bar{\bar{1}}},\quad \bar{\bar{3}}+\bar{2}+\underline{\bar{\bar{1}}},\quad 3+\bar{\bar{2}}+\underline{\bar{\bar{1}}},\quad \bar{3}+\bar{\bar{2}}+\underline{\bar{\bar{1}}},\quad \bar{\bar{3}}+\bar{\bar{2}}+\underline{\bar{\bar{1}}},$$

$$\underline{2}+\underline{2}+\underline{2},\quad \underline{\bar{2}}+\underline{2}+\underline{2},\quad \underline{\bar{\bar{2}}}+\underline{2}+\underline{2},\quad 3+\underline{1}+\underline{1}+\underline{1},\quad \bar{3}+\underline{1}+\underline{1}+\underline{1}$$

$$\bar{\bar{3}}+\underline{1}+\underline{1}+\underline{1},\quad 3+\underline{\bar{1}}+\underline{1}+\underline{1},\quad \bar{3}+\underline{\bar{1}}+\underline{1}+\underline{1},\quad \bar{\bar{3}}+\underline{\bar{1}}+\underline{1}+\underline{1},\quad 3+\underline{\bar{\bar{1}}}+\underline{1}+\underline{1},$$

$$\bar{3}+\bar{\bar{1}}+\underline{1}+\underline{1},\quad \bar{\bar{3}}+\underline{\bar{\bar{1}}}+\underline{1}+\underline{1},\quad 2+2+\underline{1}+\underline{1},\quad \bar{2}+2+\underline{1}+\underline{1},\quad \bar{\bar{2}}+2+\underline{1}+\underline{1},$$

$$2+2+\bar{\underline{1}}+\underline{1},\ \bar{2}+2+\bar{\underline{1}}+\underline{1}, \bar{\bar{2}}+2+\bar{\underline{1}}+\underline{1},\ 2+2+\bar{\bar{\underline{1}}}+\underline{1},\ \bar{2}+2+\bar{\bar{\underline{1}}}+\underline{1},$$

$$\bar{\bar{2}}+2+\bar{\bar{\underline{1}}}+\underline{1},\ 2+\underline{1}+\underline{1}+\underline{1}+\underline{1},\ \bar{2}+\underline{1}+\underline{1}+\underline{1}+\underline{1},\ \bar{\bar{2}}+\underline{1}+\underline{1}+\underline{1}+\underline{1}, 2+\bar{\underline{1}}+\underline{1}+\underline{1}+\underline{1},$$

$$\bar{2}+\bar{\underline{1}}+\underline{1}+\underline{1}+\underline{1}, \bar{\bar{2}}+\bar{\underline{1}}+\underline{1}+\underline{1}+\underline{1},\ 2+\bar{\bar{\underline{1}}}+\underline{1}+\underline{1}+\underline{1},\ \bar{2}+\bar{\bar{\underline{1}}}+\underline{1}+\underline{1}+\underline{1},\ \bar{\bar{2}}+\bar{\bar{\underline{1}}}+\underline{1}+\underline{1}+\underline{1},$$

$$\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1},\ \bar{\underline{1}}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1},\ \bar{\bar{\underline{1}}}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}.$$

In the above table, 1 is under lined in each partition to specify its least property. The over partitions have to be counted taking into consideration the over lines on the parts. The number of such over partition is 225and therefore the sum of the least parts (each being 1) is 225.

Acknowledgement: The authors are thankful to (Rtd) professor I.Ramabhdra sarma for his valuable suggestions and comments during preparation of this paper.

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