Second Order Triple Fibonacci Sequences and Some Properties

Kiran Singh Sisodiya[1], Vandana Gupta[2], V.H.Badshah[3]

Sobhasaria Group of Institution, Sikar (Rajasthan)[1]
Govt. Kalidas Girls College, Ujjain,(M.P.)India[2]
Department of Mathematics, Vikram University, Ujjain,(M.P.)India[3]

***Abstract*: - Fibonacci sequences stands as a kind of super sequence with fabulous properties. This note presents Fibonacci –Triple sequences that may also be called 3-F sequences. This is the explosive development in the region of Fibonacci sequence. Our purpose of this paper is to demonstrate fundamental properties of Fibonacci-Triple sequence.**

***2010 Mathematics Subject Classifications*: 11B50, 11B99**

***Key Words and Phrases*: Fibonacci Sequence, Fibonacci like sequence, Fibonacci –Triple Sequences.**

I. INTRODUCTION

Fibonacci –Triple sequence is the explosive development in the region of Fibonacci sequence. The Fibonacci –Triple Sequence is a new direction in generalization of coupled Fibonacci sequences. Much work has been done to study on Fibonacci-Triple sequence. The concept of Fibonacci-Triple sequence was first introduced by J.Z. Lee and J.S. Lee [1].He described new ideas for Fibonacci-Triple sequences and called 3-Fibonacci sequences or 3-F sequences. The Fibonacci-Triple sequence involves three sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa.

In this paper we present basic concepts that will be used to construct Fibonacci-Triple sequences of second order with fascinating properties. At present, we shall describe two specific schemes for 3-F sequence and fundamental properties with proof.

II. TRIPLE -FIBONACCI SEQUENCE OF SECOND ORDER

Let $(\alpha_i)_{i=0}^{\infty}$ $(\beta_i)_{i=0}^{\infty}$ and $(\gamma_i)_{i=0}^{\infty}$ be three infinite sequences and six arbitrary real numbers a, b, c, d, e and f be given. Then J.Z. Lee and J.S. Lee [1]defined following specific scheme for three sequences and derived recurrent formula:

First Scheme:

$$\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \alpha_1 = d, \beta_1 = e, \gamma_1 = f$$

$$\alpha_{n+2} = \beta_{n+1} + \beta_n \qquad n \geq 0$$

$$\beta_{n+2} = \gamma_{n+1} + \gamma_n \qquad n \geq 0$$

….(2.1)

$$\gamma_{n+2} = \alpha_{n+1} + \alpha_n \qquad n \geq 0$$

K. Atanassov [3] considers the following specific scheme:

K.T. Atanassov [2],[3] notified 36 different schemes of 3-F sequences .There are ten schemes are trivial 3-F Sequences because they having at least one resulting sequence same as Fibonacci sequence. The twenty six remaining schemes are essential generalization of the Fibonacci sequence. These sequence have arranged in seven groups and called as the Fibonacci sequence. These sequences have arranged in seven groups and called as basic 3-F Sequences.

III. PROPERTIES OF FIRST SCHEME

First few terms of First scheme (1) are as under:

n	α_n	β_n	γ_n
0	a	b	c
1	d	e	f
2	b+ e	c+f	a+d
3	c+e+f	a+d+f	b+d+e
4	a+c+d+2f	a+b+2d+e	b+c+2e+f
5	2a+b+3d+e+f	2b+c+d+3e+f	a+2c+d+e+3f
6	a+3b+c+3d+4e+f	a+3c+b+d+3e+3f	3a+b+c+4d+e+3f

If we set a=b=c and d=e=f, then the sequences $\{\alpha_i\}_{i=0}^{\infty}$ $\{\beta_i\}_{i=0}^{\infty}$ and $\{\gamma_i\}_{i=0}^{\infty}$.will be coincide with each other and with the sequence $\{F_i\}_{i=0}^{\infty}$, which is called a generalized Fibonacci sequence, where

$$F_0(a,d) = a,\ F_1(a,d) = d,$$
$$F_{n+2}(a,d) = F_{n+1}(a,d) + F_n(a,d)$$

Now we present fundamental properties of First schemes.

Theorem 1(a) For ever integer $n \geq 0$.

(a) $\alpha_{n+6} = 8\alpha_{n+1} + 5\alpha_n$.

(b) $\beta_{n+6} = 8\beta_{n+1} + 5\beta_n$.

(c) $\gamma_{n+6} = 8\gamma_{n+1} + 5\gamma_n$.

Proof. (a)To prove this, we shall use induction method. If n=0 then

$$\alpha_6 = \beta_5 + \beta_4 = (\gamma_4 + \gamma_3) + .(\gamma_3 + \gamma_2)$$

By Scheme(2.1)

$$= \alpha_3 + \alpha_2 + \alpha_2 + \alpha_1 + \alpha_2 + \alpha_1 + \alpha_1 + \alpha_0$$

By.Scheme(2.1)

$$= \alpha_3 + 3\alpha_2 + 3\alpha_1 + \alpha_0$$

By Scheme(2.1)

$$= \alpha_3 + 3(\beta_1 + \beta_0) + 3\alpha_1 + \alpha_0$$

By.Scheme(2.1)

$$= \beta_2 + \beta_1 + 3\beta_1 + 3\beta_0 + 3\alpha_1 + \alpha_0$$

By.Scheme(2.1)

$$= \gamma_1 + \gamma_0 + \beta_1 + 3\beta_1 + 3\beta_0 + 3\alpha_1 + \alpha_0$$

By.Scheme(2.1)

$$= (\gamma_1 + \beta_1 + 3\beta_1 + 3\alpha_1) + (3\beta_0 + \gamma_0 + \alpha_0)$$

By hypothesis

$$= (\alpha_1 + \alpha_1 + 3\alpha_1 + 3\alpha_1) + (3\alpha_0 + \alpha_0 + \alpha_0)$$

By Scheme (2.1)

$$\alpha_6 = 8\alpha_1 + 5\alpha_0$$

Let us assume that the result is true for some integer's $n \geq 1$ .Then

$$\alpha_{n+7} = \beta_{n+6} + \beta_{n+5}$$

By Scheme (2.1)

$$= \gamma_{n+5} + \gamma_{n+4} + \gamma_{n+4} + \gamma_{n+3}$$

By Scheme (2.1)

$$= \alpha_{n+4} + \alpha_{n+3} + 2(\alpha_{n+3} + \alpha_{n+2}) + \alpha_{n+2} + \alpha_{n+1}$$

By Scheme (2.1)

$$= \alpha_{n+4} + 3\alpha_{n+3} + 3\alpha_{n+2} + \alpha_{n+1}$$

By Scheme (2.1)

$$= \beta_{n+3} + \beta_{n+2} + 3(\beta_{n+2} + \beta_{n+1}) + 3(\beta_{n+1} + \beta_n) + \alpha_{n+1}$$

By Scheme (2.1)

$$= \gamma_{n+2} + \gamma_{n+1} + \beta_{n+2} + 3\beta_{n+2} + 3\beta_{n+1} + 3\alpha_{n+2} + \alpha_{n+1}$$

By Scheme(2.1)

$$= (\gamma_{n+2} + \beta_{n+2} + 3\beta_{n+2} + 3\alpha_{n+2}) + (3\beta_{n+1} + \gamma_{n+1} + \alpha_{n+1})$$

BY hypothesis

$$= (\alpha_{n+2} + \alpha_{n+2} + 3\alpha_{n+2} + 3\alpha_{n+2}) + (3\alpha_{n+1} + \alpha_{n+1} + \alpha_{n+1})$$

$$\alpha_{n+7} = 8\alpha_{n+2} + 5\alpha_{n+1}$$

Hence the result is true for all integers $n \geq 0$

Similar proofs can be given for remaining parts (b) and (c).

Theorem 2.For every integer $n \geq 0$,

$$\text{a) } \alpha_n = \begin{cases} F_{n-1}a + F_n f, if n \equiv 0 (\bmod 3) \\ F_{n-1}b + F_n d, if n \equiv 1 (\bmod 3) \\ F_{n-1}c + F_n e, if n \equiv 2 (\bmod 3) \end{cases}$$

$$\text{b) } \beta_n = \begin{cases} F_{n-1}b + F_n d, if n \equiv 0 (\bmod 3) \\ F_{n-1}c + F_n e, if n \equiv 1 (\bmod 3) \\ F_{n-1}a + F_n f, if n \equiv 2 (\bmod 3) \end{cases}$$

$$\text{c) } \gamma_n = \begin{cases} F_{n-1}c + F_n e, if n \equiv 0 (\bmod 3) \\ F_{n-1}a + F_n f, if n \equiv 1 (\bmod 3) \\ F_{n-1}b + F_n d, if n \equiv 2 (\bmod 3) \end{cases}$$

Proof.(a)By Theorem (1),we have

$$\alpha_{n+6} = 8\alpha_{n+1} + 5\alpha_n \qquad \text{for } n \geq 0$$

When $n \equiv 0$ (mod 3), assume n = 3m,we have

$$\alpha_{3(n+2)} = 8\alpha_{3(m+1)} + 5\alpha_{3m} \text{ with } \alpha_0 = a,$$

$$\alpha_3 = a + d + f$$

Let $G_m = \alpha_{3m}$,we have

$$G_{m+2} = 8G_{m+1} + 5G_m, \text{ with}$$

$$G_0 = a, G_1 = \text{c + e + f}$$

Therefore, we get

$$G_m = \frac{(\sqrt{5}-1)a + 2f}{2\sqrt{5}}(2+\sqrt{5})^m + \frac{(\sqrt{5}-1)a - 2f}{2\sqrt{5}}(2-\sqrt{5})^m$$

$$G_m = \frac{1}{\sqrt{5}}[a\{\frac{1+\sqrt{5}}{2}\}^{3m-1} - (\frac{1-\sqrt{5}}{2})^{3m-1}\} + f\{(\frac{1+\sqrt{5}}{2})^{3m} - (\frac{1-\sqrt{5}}{2})^{3m}\}]$$

Since $(\frac{1+\sqrt{5}}{2})^3 = 2 \pm \sqrt{5}$

$$= F_{n-1}a + F_n f$$

i.e. $\alpha_n = F_{n-1}a + F_n f$

Using same process to get:

$$\alpha_n = \begin{cases} F_{n-1}a + F_n f, if n \equiv 0(\bmod 3) \\ F_{n-1}b + F_n d, if n \equiv 1(\bmod 3) \\ F_{n-1}c + F_n e, if n \equiv 2(\bmod 3) \end{cases}$$

Similar proofs can be given for remaining parts (b) and (c).

Theorem 3.For every integer $n \geq 0$

$$\sum_{k=0}^{n}(\alpha_k + \beta_k + \gamma_k) = F_{n+1}(\alpha_0 + \beta_0 + \gamma_0) + (F_{n+2} - 1)(\alpha_1 + \beta_1 + \gamma_1)$$

Proof: To prove this, we shall use induction method.

If n=0 assume that the result is true for some integer $n \geq 1$, then

$$\sum(\alpha_k + \beta_k + \gamma_k) = (\alpha_{n+1} + \beta_{n+1} + \gamma_{n+1}) + \sum_{k=0}^{n}(\alpha_k + \beta_k + \gamma_k)$$

$$= (\alpha_{n+1} + \beta_{n+1} + \gamma_{n+1}) + \sum_{k=0}^{n}(\alpha_k + \beta_k + \gamma_k)$$

$$= (\alpha_{n+1} + \beta_{n+1} + \gamma_{n+1}) + F_{n+1}(\alpha_0 + \beta_0 + \gamma_0) + (F_{n+2} - 1)(\alpha_1 + \beta_1 + \gamma_1)$$

(By indu. Hypo.)

$$= F_n(\alpha_0 + \beta_0 + \gamma_0) + F_{n+1}(\alpha_1 + \beta_1 + \gamma_1) + (F_{n+1})(\alpha_0 + \beta_0 + \gamma_0) + (F_{n+2} - 1)(\alpha_1 + \beta_1 + \gamma_1)$$

$$\sum_{k=0}^{n}(\alpha_k + \beta_k + \gamma_k) = F_{n+2}(\alpha_0 + \beta_0 + \gamma_0) + (F_{n+3} - 1)(\alpha_1 + \beta_1 + \gamma_1)$$

Hence the result is true for all integers $n \geq 0$.

Theorem 4: For every integer $n \geq 0$

$$(\alpha_{n+2} + \beta_{n+2} + \gamma_{n+2}) = \sum_{k=0}^{n} (\alpha_0 + \beta_0 + \gamma_0) + (\alpha_1 + \beta_1 + \gamma_1)$$

Proof can be given by induction method.

REFERENCES

[1]. J.Z. Lee and J.S. Lee , Some properties of the generalization of the Fibonacci sequence, The Fibonacci Quarterly,25(1987),111-117
[2]. K. Atanassov , On a generalization of the Fibonacci sequence in the case of three sequences,The Fibonacci Quarterly 27(1989),7-10.
[3]. K. Atanassov , V.Atanassov, A.Shannon and J.Turner. New Visual Perspectives on Fibonacci Numbers, World Scientific,2002.
[4]. Hirschhorn M., Non-trivial intertwined second-orderrecurrence relation. The Fibonacci Quartarly, Vol. 43 (2005), No. 4, 316-325.
[5]. Hirschhorn, Coupled second-order recurrences. Fibonacci sequence, The Fibonacci Quartarly, Vol. 44 (2006), No. 1, 20-25.
[6]. Hirschhorn, Coupled third-order recurrences. Fibonacci sequence, The Fibonacci Quarterly, Vol. 44 (2006), No. 1, 26-31.
[7]. Lee J.-Z., J.-S. Lee, Some properties of the generalization of the Fibonacci sequence.The Fibonacci Quarterly,Vol. 25 (1987) No. 2, 111-117.
[8]. Randic M., D. Morales, O. Araujo, Higher-order Fibonacci numbers, Journal of Mathematical Chemistry,1996, Vol.20, No.1-2, 79-94.
[9]. Shannon A., R. Melham, Carlitz generalizations of Lucas and Lehmer sequences,The Fibonacci Quartarly, Vol. 31 (1993), No. 2, 105-111.
[10]. Spickerman W., R. Creech, The (2;T) generalized Fibonacci sequences,The Fibonacci Quartarly,Vol. 35 (1997), No. 4, 358-360.