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On Specific Properties Common to a Graph and its Complement

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Abstract **– Computing the relation between the chromatic number and the independence number of a graph and its complement respectively. We investigate this relation for some specific classes of graphs such as cycles, paths, stars and wheel graphs. Also for any graph having the same independence number for a graph and its complement, we find its bounds with the order of the graph.**

Keywords **– Chromatic number, Complement, Independence number, Order, Perfect graph.**

I. INTRODUCTION

We consider only finite, loopless graphs without multiple edges and the graph may be connected or disconnected. Thus, we adopt the notation $G = (V, E)$, where V is the finite vertex set of the *graph G* and the edge set E is a specified set of 2-element subsets of V. We denote $|V| = p$ and $|E| = q$ where $|V|$ and $|E|$ are the order and size respectively of the graph G. The *complement* of G is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ is the complement of E in the set of all 2-element subsets of V. The *degree* of a vertex v in a graph G, denoted by *d(v)* or *deg(v)*, is the number of edges incident to v. If $\mathrm{U} = \{\mathrm{v}_1, \mathrm{v}_2, \ldots \mathrm{v}_k\} \subseteq V$ is such that no $v_i v_j \in E$, U is called an *independent set*. A maximum independent set is called the *independence number* of G and is denoted by α. A *clique* is a maximal complete subgraph, and the maximum size (number of vertices) of a clique is the clique number of G, denoted by ω. The *chromatic number* χ of G is the minimum number of colors necessary to color the vertices of G such that no two adjacent vertices are colored alike. A *perfect* graph is a graph in which the chromatic number of every induced subgraph equals the clique number of that subgraph. An induced cycle of odd length at least 5 is called an *odd hole* and an induced subgraph that is the complement of an odd hole is called an *odd antihole*. A graph that does not contain any odd holes or antiholes is called a *Berge* graph.

The graph and its complement have many properties in common. The study of the specific properties common to a graph and its complement was introduced first by Jin Akiyama and Frank Harary in 1979 [1]. They investigated the conditions under which both graph G and its complement possess a specified property. In particular, they characterized all graphs G in [1] for which G and $\overline{G}$ both (a) have connectivity one, (b) have line-connectivity one, (c) are 2 connected, (d) are forests, (e) are bipartite, (f) are outerplanar and (g) are Eulerian. The other work of Jin Akiyama and Frank Harary includes the study of a graph and its complement with specific properties such as girth and circumference, self-complementary graph, chromatic number etc. They studied the conditions when a graph and its complement have the same girth and circumference [2]. In [3] Jin Akiyama and Frank Harary studied the results of counting self-complementary graphs. In this series of results, they characterized all the graphs G such that both G and its complements have the same number of end vertices which can be 0, 1 or 2. The relation between the independence number and the chromatic number was generalized by E. Sampath Kumar in 1992 [4]. Most of the work concentrates in relation to the graph and its complement on only one property. In this paper we look into two properties namely the independence number and the chromatic number of a graph and its complement. In the later part we discuss the relation between the independence number and the order.

II. RELATION BETWEEN INDEPENDENCE NUMBER AND CHROMATIC NUMBER OF A GRAPH AND ITS COMPLEMENT

Theorem 2.1 If G or $\overline{G}$ is C_{2n+1} graph where n = 2, 3, 4…, then $\alpha(G) \neq \chi(\overline{G})$.

Proof: Let G be a graph such that G or $\overline{G}$ is C_{2n+1} where n = 2, 3, 4…. If $G = C_{2n+1}$, where n = 2, 3, 4.., then we have, $\alpha(G) = n$. Since G is a cycle, $\chi(\overline{G}) = n+1$. Thus $\alpha(G) \neq \chi(\overline{G})$. If $\overline{G}$ is C_{2n+1} where n = 2, 3, 4…, $\overline{G}$ is an odd cycle of length at least 5 and $\chi(\overline{G}) = 3$. Since the independence number of any graph is equal to the clique number of its complement and $\overline{G}$ is an odd cycle, the maximum clique number of $\overline{G}$ is 2 since $n \geq 2$. This implies, $\alpha(G) = 2$. Therefore $\alpha(G) \neq \chi(\overline{G})$.

Corollary 2.2 *If G is a graph such that G or* $\overline{G}$ *has a girth at least 4 then* $\alpha(G) \neq \chi(\overline{G})$.

Proof: Let G be a graph such that G and $\overline{G}$ have girth at least 4. By [2], we have there are no graphs G other than C_5 such that G and $\overline{G}$ have girth at least 4. Hence, G is C_5. By theorem 2.1 we have, $\alpha(G) \neq \chi(\overline{G})$.

Proposition 2.3 *For any Graph G if G is perfect then* $\chi(G) = \alpha(\overline{G})$

Proof: Let G be perfect then by the definition $\chi(G) = \omega(G)$. We know that for any graph G the independence number $\alpha(G)$ is always equal to the clique number of its complement $\omega(\overline{G})$, i.e., If $\omega(G)$ is the clique number of the graph G and $\alpha(\overline{G})$ is the independence number of its complement $\overline{G}$ the we have, $\omega(G) = \alpha(\overline{G})$. Therefore $\chi(G) = \alpha(\overline{G})$.

Proposition 2.4 *For any Graph G if G is Berge then* $\chi(G) = \alpha(\overline{G})$.

Proof: Let G be a Berge graph. By Strong Perfect Graph Theorem [5], we have, a graph is perfect if and only if it is Berge. Hence by proposition 2.3 we have, $\chi(G) = \alpha(\overline{G})$.

Proposition 2.5 *For any path* P_n *of order n,* $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil = \chi(\overline{P}_n)$.

Proof: Let $\alpha(P_n)$ be the independence number of the path P_n of order n then, $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$. Also $\chi(\overline{P}_n) = \left\lceil \frac{n}{2} \right\rceil$. Hence for any path P_n of order n, $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil = \chi(\overline{P}_n)$.

Proposition 2.6 P_0, P_2, P_3 *are the only paths among all* P_n *($n \geq 1$), such that* $\alpha(P_n) = \alpha(\overline{P}_n)$.

Proof: For any path P_n when $n \geq 5$, $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$. Thus, when $n \geq 5$ we have, $\alpha(P_n) \geq 3$. Also, the clique number for any P_n is 2. Hence, the independence number of its complement can never be greater than 2. i.e., $\alpha(\overline{P}_n) = 1$, when $n = 0$, and $\alpha(\overline{P}_n) = 2$ otherwise. Hence when $n \geq 5$, $\alpha(P_n) \neq \alpha(\overline{P}_n)$. Also, when $n < 5$, only P_2, P_3 have $\alpha(P_n) = \alpha(\overline{P}_n) = 2$ and for P_0, $\alpha(P_n) = \alpha(\overline{P}_n) = 1$. Thus P_0, P_2, P_3 are the only paths among all P_n where $n \geq 1$, such that $\alpha(P_n) = \alpha(\overline{P}_n)$.

Proposition 2.7 *If G is a star* S_n*, where n is the order of G, then* $\alpha(S_n) = \chi(\overline{S}_n) = n-1$.

Proof: Let S_n be a star of order n. By the definition of star, it is clear that every star is a complete bipartite graph $K_{1,n-1}$. Thus $\alpha(S_n) = n-1$. Also by the definition of a star we have, $\overline{S}_n$ is a complete graph with $n-1$ vertices and an isolated vertex, i.e. $K_1 \cup K_{1,n-1}$. Therefore $\chi(\overline{S}_n) = n-1$. Thus $\alpha(S_n) = \chi(\overline{S}_n) = n-1$.

Theorem 2.8 *For any wheel* W_n *of order n,*

(i) $\alpha(W_n) = \chi(\overline{W}_{n-1})$, *where* $\overline{W}_{n-1}$ *is the complement* W_{n-1}, *when* $n \geq 6$.

(ii) $\chi(W_n) = \alpha(\overline{W}_{n-1}) = 3$ *and*

$$\alpha(W_n) = \chi(\overline{W}_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor, \text{ when } n \text{ is odd.}$$

Proof: Consider a wheel W_n of order n.

Case (i): when $n \geq 6$,

Clearly the independence number of any n-cycle is $\left\lfloor \frac{n}{2} \right\rfloor$. Since the maximum induced cycle contained in a wheel W_n is $(n-1)$, we have, $\alpha(W_n) = \left\lfloor \frac{n-1}{2} \right\rfloor$. Since $\overline{W}_n = \overline{C}_{n-1} \cup K_1$, we have, $\chi(\overline{W}_n) = \chi(\overline{C}_{n-1})$. Also for any odd cycle of length at least 5, $\chi(\overline{C}_{2n+1}) = n+1$. When $n = \frac{n-2}{2}$ we have, $\chi(\overline{C}_{n-1}) = \frac{n}{2}$. Therefore, $\chi(\overline{W}_n) = \left\lfloor \frac{n}{2} \right\rfloor$ implies $\chi(\overline{W}_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor$. Hence $\alpha(W_n) = \chi(\overline{W}_{n-1})$

Case (ii): when n is odd,

By the definition of a wheel we have, a wheel graph W_n is a graph with n vertices, formed by connecting a single vertex to all vertices of a $(n-1)-$cycle. Hence when n is odd, W_n contains induced cycles which are either an even cycle or C_3 i.e. W_n does not contain an induced odd cycle whose length is at least 5 or a complement of one. Therefore $\chi(W_n) = \alpha(\overline{W}_n)$. Since $\alpha(\overline{W}_n) = 3$ we have, $\chi(W_n) = \alpha(\overline{W}_{n-1}) = 3$.

We know that the independence number of any wheel is equal to the independence number of the maximum induced cycle contained in it, also independence number or any n-cycle is $\left\lfloor \frac{n}{2} \right\rfloor$ when n is even. Therefore, $\alpha(W_n) = \chi(\overline{W}_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor$.

III. RELATION BETWEEN INDEPENDENCE NUMBER OF A GRAPH AND ITS COMPLEMENT AND THE ORDER OF THE GRAPH

Let us define a nK_n graph to be a collection of n number of complete graphs of order n. Clearly the order of nK_n is n^2. Figure 3.1 represents $3K_3$ and its complement respectively.

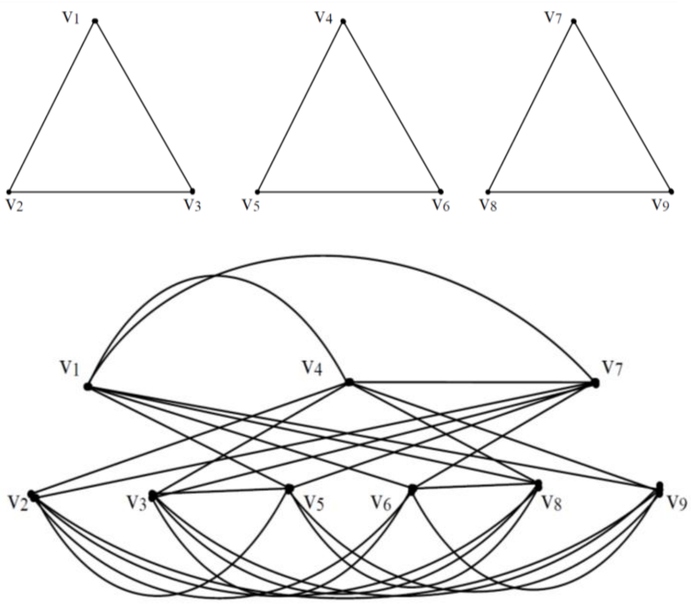

Figure 3.1

Consider a graph such that both *G* and its complement have the same independence number. We can note that other than for C_5 where $|V(G)| = n^2 + 1$, all other graphs have $2n - 1 \le |V(G)| \le n^2$, where *n* is the independence number of the graph. Since the clique number of nK_n is *n*, clearly the independence number of nK_n graph is equal to the independence number of its complement.

Theorem 3.1 *A graph G is an nK_n graph if and only if $\bar{G}$ is $(p-n)-$regular and $\alpha(G) = \alpha(\bar{G}) = n$, where p is the order of G and where $n \ge 1$.*

Proof: Assuming that *G* is an nK_n graph of order *p*, where $n \ge 1$ we have, *G* is a collection of *n* number of K_n. Therefore, *G* has clique number *n*. Also, we know that $\omega(G) = \alpha(\overline{G})$. Thus, we have $\alpha(\overline{G}) = n$. Since the independence number of any complete graph is 1 and *G* is a collection of *n* number of K_n, hence $\alpha(G) = n$. Therefore $\alpha(G) = \alpha(\overline{G}) = n$. Also, the vertices in *G* are adjacent to $(n - 1)$ vertices, thus every vertex in $\overline{G}$ is adjacent to $(p-n)$ vertices. Thus $\overline{G}$ is $(p-n)-$regular graph. Hence if *G* is an nK_n graph where $n \ge 1$, then $\bar{G}$ is $(p-n)-$regular and $\alpha(\overline{G}) = n$ where *p* is the order of *G*. Conversely, suppose that *G* is a graph such that $\overline{G}$ is $(p-n)-$regular and $\alpha(G) = \alpha(\overline{G}) = n$ where *p* is the order of *G*. Since $\overline{G}$ is regular graph, *G* is also a regular graph. Let us assume that *G* is *m*–regular graph where $0 \le m \le p - 1$. Also, if *p* is the order of *G* then, the sum of the degree of a vertex of any graph and its complement is p−1. Since $\overline{G}$ is $(p-n)-$regular and *G* is *m*-regular we have, $m + p - n = p - 1$. Therefore $m = n - 1$. Thus, *G* is an $n - 1$ regular graph. Given $\alpha(G) = \alpha(\overline{G}) = n$, we have *G* and $\overline{G}$ has *n* number of vertices which are pairwise adjacent and *n* number of vertices which are pairwise non-adjacent. Since *G* is $n - 1$ regular, we have every vertex in *G* are adjacent to $n-1$ vertices which together form a complete graph of order *n*. Since *n* number of vertices are pairwise non-adjacent, each vertex in the independence set of *G* is a vertex in a complete graph K_n. Thus, clearly *G* is a collection of n number of K_n. Hence *G* is nK_n.

Corollary 3.2 *For any graph G, if $\alpha(G) = \alpha(\overline{G}) = n$ where n ≥ 1 then,*

(i) $|V(G)| = n^2 + 1$, when G is C_5.

(ii) $2n-1 \le |V(G)| \le n^2$ otherwise, where $|V(G)|$ is the order of G.

Proof: Let G be a graph such that $\alpha(G) = \alpha(\overline{G}) = n \ge 1$.

Case (i): When *G* is C_5.

The result is obvious since C_5 is a self-complementary graph, clearly $\alpha(G) = \alpha(\overline{G}) = 2$. Also, the $|V(G)| = 5 = n^2 + 1$, where n is the independence number of *G*.

Case (ii): When *G* is not C_5.

Since $\alpha(G) = \alpha(\overline{G}) = n$, we have *G* should have at most *n* vertices which are pairwise non-adjacent and at most *n* vertices which are pairwise adjacent. Also, there exists a vertex v such that $v \in \alpha(G)$ and $v \in \alpha(\overline{G})$.

Hence $|V(G)| \ge n + n - 1 = 2n - 1$. Let us assume that *p* is the order of *G*. i.e. $|V(G)| = p$. Since $\alpha(G) = \alpha(\overline{G}) = n$, G should have at most *n* vertices which are pairwise non-adjacent and at most *n* vertices which are pairwise adjacent. Also, every vertex in *G* or $\bar{G}$ should have at most n vertices which are non-adjacent. Thus, the maximum degree of G is or $\overline{G}$ is $p - n$. Without loss of generality let us assume that *G* has maximum degree for all p vertices. i.e., G is a $(p-n)-$regular graph. Also, we know that $\alpha(G)=\alpha(\overline{G})=n$. Hence by theorem 3.1 $\bar{G}$ is a nK_n graph which is of order n^2. Hence for any graph with $\alpha(G)=\alpha(\overline{G})=n$ has order n^2.

Corollary 3.3 *For any graph G,* $\alpha(G)=\alpha(\overline{G})=2$ *if and only if G is any one of the following graph* C_4, C_5, P_3 *and* P_4 *or a complement of one.*

Proof: Let us assume that G or $\overline{G}$ is any one of the following graph C_4, C_5, P_3 and P_4. Clearly by computation we have $\alpha(G)=\alpha(\overline{G})=2$. Conversely, suppose that $\alpha(G)=\alpha(\overline{G})=2$, by theorem 3.2 we have, for any graph G, if G and $\overline{G}$ has independence number n then the order of G is $(2n-1)\le|V(G)|\le n^2$. Since $\alpha(G)=\alpha(\overline{G})=2$, clearly the order of G is at least 3 and at most 5. Thus, among all graphs with order at least 3 and at most 5, C_4, C_5, P_3 and P_4 or its complement have $\alpha(G)=\alpha(\overline{G})=2$.

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