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Inverse Transient Thermoelastic Problem of Semi-Infinite Rectangular Plate

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***Abstract*- This paper is concerned with transient thermoelastic problem in which we need to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of semi-infinite rectangular plate when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.**

***Key Words:* Semi-infinite rectangular plate, transient problem, Integral transform, inverse problem**

I. INTRODUCTION

In 1999, **Adams and Bert [1]** studied thermoelastic vibrations of a laminated rectangular plate subjected to a thermal shock. **Tanigawa and Komatsubara [2]** discussed thermal stress analysis of a rectangular plate and its thermal stress intensity factor for compressive stress field. **Vihak; Yuzvyak and Yasinskij [3]**: derived the solution of the plane thermoelasticity problem for a rectangular domain. **Dange; Khobragade and Durge [4]** studied three dimensional inverse transient thermoelastic problem of a thin rectangular plate. **Ghume and Khobragade [5]** investigated deflection of a thick rectangular plate. **Roy and Khobragade [6]** discussed transient thermoelastic problem of an infinite rectangular slab. **Lamba and Khobragade [7]** studied thermoelastic problem of a thin rectangular plate due to partially distributed heat supply.

In 2012**, Sutar and Khobragade [8]** discussed inverse thermoelastic problem of heat conduction with internal heat generation for the rectangular plate. **Khobragade; Hiranwar;** and Khalsa [9] derived thermal deflection of a thick clamped rectangular plate. **Roy; Bagade and Khobragade [10]** studied thermal stresses of a semi infinite rectangular beam. **Jadhav and Khobragade [11]** discussed inverse thermoelastic problem of a thin finite rectangular plate due to internal heat source. **Singru and Khobragade [12]** studied thermal stress analysis of a thin rectangular plate with internal heat source. **Further Singru and Khobragade [13] derived**, Thermal stresses of a semi-infinite rectangular slab with internal heat generation.

In this paper, an attempt has been made to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of semi-infinite rectangular plate occupying the space D: $0 \le x \le a$, $0 \le y \le \infty$ with the boundary conditions that the temperature is maintained at zero on the edges $y = 0, \infty$ and on the edge x = o of a thin rectangular plate respectively.

II. STATEMENT OF THE PROBLEM

Consider semi-infinite rectangular plate occupying the space D : $0 \le x \le a$, $0 \le y \le \infty$. The displacement components u_x and u_y in the x and y- direction represented in the integral form as [2] are

$$u_x = \int \left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \tag{2.1}$$

$$u_y = \int \left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \tag{2.2}$$

ν and α are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate respectively and U(x,y, t) is the Airy's stress function which satisfy the following relation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 U = -\alpha E \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T \tag{2.3}$$

where E is the Young's modulus of elasticity and T is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{2.4}$$

subject to the initial condition

$$T(x, y, 0) = 0 \tag{2.5}$$

the boundary conditions

$$T(0, y, t) = 0 \tag{2.6}$$

$$T(a, y, t) = g(y, t) \text{ (unknown)} \tag{2.7}$$

$$T(x,0,t) = 0 \tag{2.8}$$

$$T(x,\infty,t) = 0 \tag{2.9}$$

the interior condition

$$T(\xi, y, t) = f(y,t)\ ,\ 0 < \xi < a \quad \text{(known)} \tag{2.10}$$

where k is the thermal diffusivity of the material of the plate.

The stress components in terms of U are given by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \tag{2.11}$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \tag{2.12}$$

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \tag{2.13}$$

Equations (2.1) to (2.13) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying Fourier sine transform to the equations (2.4), (2.5), (2.6), (2.7) and (2.10) and using (2.8), (2.9) one obtains

$$\frac{d^2\overline{T}_s}{dx^2} - p^2\overline{T}_s = \frac{1}{k}\frac{d\overline{T}_s}{dt} \tag{3.1}$$

where $p^2 = m^2\pi^2$ (3.2)

$$\overline{T}_s(x,m,0) = 0 \tag{3.3}$$

$$\overline{T}_s(0,m,t) = 0 \tag{3.4}$$

$$\overline{T}_s(a,m,t) = \overline{g}_s(m,t) \tag{3.5}$$

$$\overline{T}_s(\xi,m,t) = \overline{f}_s(m,t) \tag{3.6}$$

where $\overline{T}_s$ denotes Fourier sine transform of T and m is sine transform parameter.

Applying Laplace transform to the equations (3.1), (3.4), (3.5), (3.6) and using (3.3) one obtains

$$\frac{d^2\overline{T}^*_s}{dx^2} - q^2\overline{T}^*_s = 0 \tag{3.7}$$

where $q^2 = p^2 + \frac{s}{k}$ (3.8)

$$\overline{T}^*_s(0,m,s) = 0 \tag{3.9}$$

$$\overline{T}^*_s(a,m,s) = \overline{g}^*_s(m,s) \tag{3.10}$$

$$\overline{T}^*_s(\xi,m,s) = \overline{f}^*_s(m,s) \tag{3.11}$$

where $\overline{T}^*_s$ denotes Laplace transform of $\overline{T}_s$ and s is Laplace transform parameter.

Equation (3.6) is a second order differential equation whose solution gives

$$\overline{T}^*_s(x,m,s) = Ae^{qx} + Be^{-qx} \tag{3.12}$$

where A, B are arbitrary constants.

Using (3.9) and (3.11) in (3.12) one obtains

$$A + B = 0 \tag{3.13}$$

$$Ae^{q\xi} + Be^{-q\xi} = \overline{f}_s(m,s) \tag{3.14}$$

Solving (3.13) and (3.14) one obtains

$$A = \frac{\overline{f}^*_s(m,s)}{e^{q\xi} - e^{-q\xi}}\ , \quad B = -\frac{\overline{f}^*_s(m,s)}{e^{q\xi} - e^{-q\xi}}$$

Substituting the values of A and B in (3.12) one obtains

$$\overline{T}^*_s(x,m,s) = \overline{f}^*_s(m,s)\frac{\sinh(qx)}{\sinh(q\xi)} \tag{3.15}$$

Using the condition (3.10) to the solution (3.15) one obtains

$$\overline{g}^*_s(m,s) = \overline{f}^*_s(m,s)\frac{\sinh(qa)}{\sinh(q\xi)} \tag{3.16}$$

Applying inverse Laplace transform to the equation (3.15) one obtains

$$\overline{T}_s(x,m,t) = L^{-1}\left[\overline{f}^*_s(m,s)\frac{\sinh(qx)}{\sinh(q\xi)}\right] \tag{3.17}$$

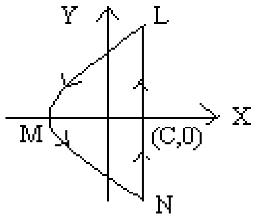
To evaluate $L^{-1}\left[\overline{f}^*_s(m,s)\overline{g}^*_1(s)\right]$

where $\overline{g}^*_1(s) = \frac{\sinh(qx)}{\sinh(q\xi)}$ (3.18)

Using inversion integral to the equation (3.18) one obtains

$$\bar{g}_1(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{\sinh(qx)}{\sinh(q\xi)}\,ds \qquad (3.19)$$

Now to calculate the inversion integral (3.19) where c is greater than the real part of all singularities of the integrand. The integral is a single valued function of s in the region bounded by the closed Bromwich contour of the figure given below:



The line NL is chosen so as to lie all the poles to the right, which are given by

$$s = s_n = \frac{in\pi}{\xi}, \qquad n = 1, 2, \ldots$$

Choosing the contour so that the curved portion LMN is an arc of the circle Γ with center at the origin and radius $R = k(n+1/2)^2(\pi/\xi)^2$, so that it will not pass through zero of sinhqξ.

The integral over the circular arc tends to zero as n → ∞.

Now sinh(qξ) = 0 gives q = (inπ/ξ) ie s = s_n = (inπ/ξ), n = 1, 2, 3, …

Therefore

Residue at s_n =

$$\lim_{s\to s_n}\left[\frac{s-s_n}{\sinh\left(\xi\sqrt{\frac{s}{k}+p^2}\right)}e^{st}\sinh\left(x\sqrt{\frac{s}{k}+p^2}\right)\right]$$

$$= \lim_{s\to s_n}\left[\frac{2k\sqrt{\frac{s}{k}+p^2}}{\xi\cosh\left(\xi\sqrt{\frac{s}{k}+p^2}\right)}\right]\left[\lim_{s\to s_n} e^{st}\sinh\left(x\sqrt{\frac{s}{k}+p^2}\right)\right]$$

$$= \frac{2kin\pi}{\xi^2\cos n\pi}\left(\frac{1}{i}\right)\sin\left(ix\left(\frac{in\pi}{\xi}\right)\right)e^{-kt\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)}$$

Hence the value of $\bar{g}_1(t)$ is given by

$$\bar{g}_1(t) = \frac{2k\pi}{\xi^2}\sum_{n=1}^{\infty}(-1)^{n+1} n\sin\left(\frac{n\pi}{\xi}\right)x e^{-kt\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)}$$

Applying the Convolution Theorem to the equation (3.17) one obtains

$$\bar{T}_S(x,m,t) = \frac{2k\pi}{\xi^2}\sum_{n=1}^{\infty}(-1)^{n+1} n\sin\left(\frac{n\pi}{\xi}\right)x \times \int_0^t \bar{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (3.20)$$

Also by using the result (3.19), the equation (3.16) gives

$$\bar{g}_S(m,t) = \frac{2k\pi}{\xi^2}\sum_{n=1}^{\infty}(-1)^{n+1} n\sin\left(\frac{n\pi}{\xi}\right)a \times \int_0^t \bar{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (3.21)$$

Applying inverse Fourier sine transform to the equations (3.20) and (3.21) one obtain the expressions for the temperature distribution T(x, y, t) and unknown temperature gradient g(y, t) as

$$T(x,y,t) = \left(\frac{2k}{\xi^2}\right)\sum_{m=1}^{\infty}\sin py\sum_{n=1}^{\infty}(-1)^{n+1} n\sin\left(\frac{n\pi}{\xi}\right)x \times \int_0^t \bar{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (3.22)$$

$$g(y,t) = \left(\frac{2k}{\xi^2}\right)\sum_{m=1}^{\infty}\sin py\sum_{n=1}^{\infty}(-1)^{n+1} n\sin\left(\frac{n\pi}{\xi}\right)a \times \int_0^t \bar{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (3.23)$$

where $\bar{f}_s(m,t) = \int_0^\infty f(y,t)\sin py\,dy$

Substituting the value of T(x,y, t) from (3. 22) in (2.3) one obtains the expression for Airy's stress function U(x,y, t) as

$$U(x,y,t) = -\frac{\alpha E}{p^2}\left(\frac{2k}{\xi^2}\right)\sum_{m=1}^{\infty}\sin py\sum_{n=1}^{\infty}(-1)^{n+1} n\sin\left(\frac{n\pi}{\xi}\right)x$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (3.24)$$

IV. DETERMINATION OF THERMOELASTIC DISPLACEMENT

Substituting the value of U(x,y, t) from (3.24) in (2.1) and (2.2) one obtains the thermoelastic displacement functions u_x and u_y as

$$u_x=-\left(\frac{2\alpha k\pi}{\xi^3}\right)\sum_{m=1}^{\infty}\sin py\sum_{n=1}^{\infty}(-1)^{n+1}n^2\sin\left(\frac{n\pi}{\xi}\right)x$$

$$\times\left[2-\frac{\nu\ n^2}{\xi^2p^2}\right]$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (4.1)$$

$$u_y=-\left(\frac{2\alpha k}{p\xi^2}\right)\sum_{m=1}^{\infty}\cos py\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\left(\frac{n\pi}{\xi}\right)x$$

$$\times\left[\frac{\pi^2n^2}{\xi p^2}-\nu+1\right]$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (4.2)$$

V. DETERMINATION OF STRESS FUNCTIONS

Using (3.24) in (2.11) , (2.12) and (2.13) the stress functions are obtained as

$$\sigma_{xx}=\alpha E\left(\frac{2k}{\xi^2}\right)\sum_{m=1}^{\infty}\sin py\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\left(\frac{n\pi}{\xi}\right)x$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (5.1)$$

$$\sigma_{yy}=\frac{\alpha E}{p^2}\left(\frac{2k\pi^2}{\xi^3}\right)\sum_{m=1}^{\infty}\sin py\sum_{n=1}^{\infty}(-1)^{n+1}n^3\sin\left(\frac{n\pi}{\xi}\right)x$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (5.2)$$

$$\sigma_{xy}=\frac{\alpha E}{p}\left(\frac{2k\pi}{\xi^2}\right)\sum_{m=1}^{\infty}\cos py\sum_{n=1}^{\infty}(-1)^{n+1}n^2\cos\left(\frac{n\pi}{\xi}\right)x$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (5.3)$$

VI. SPECIAL CASE

Set $f(y,t)=\left(\frac{(1-e^{-t})y\xi}{1+y^2}\right)$ (6.1)

Applying Fourier sine transform to the equation (6.1) one obtains

$$\overline{f}_s(m,t)=\int_0^{\infty}\left(\frac{(1-e^{-t})y\xi}{1+y^2}\right)\sin pydy$$

$$=\left(1-e^{-t}\right)\left(\frac{\pi\xi}{2}e^{-p}\right) \qquad (6.2)$$

Substituting the value of $\overline{f}_s(m,t)$ from (6.2) in the equations (3.22), (3.23), one obtains

$$T(x,y,t)=\left(\frac{\pi k}{\xi}\right)\sum_{m=1}^{\infty}e^{-p}\sin py\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\left(\frac{n\pi}{\xi}\right)x$$

$$\times\int_0^t\left(1-e^{-t'}\right)e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (6.3)$$

$$g(y,t)=\left(\frac{\pi k}{\xi}\right)\sum_{m=1}^{\infty}e^{-p}\sin py\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\left(\frac{n\pi}{\xi}\right)a$$

$$\times\int_0^t\left(1-e^{-t'}\right)e^{-k\left(p^2+\frac{n^2\pi^2}{\xi^2}\right)(t-t')}dt' \qquad (6.4)$$

VII. NUMERICAL RESULT

Set $\beta = \frac{\pi k}{\xi}$, π = 3.14, *a* = 2 m, ξ =1.5 m, t = 1 sec. and k = 0.86 in the equation (6.4) to obtain

$$\frac{g(y,t)}{\beta} = \sum_{m=1}^{\infty} e^{-p} \sin py \sum_{n=1}^{\infty} (-1)^{n+1} n \sin(4.2n)$$

$$\times \int_{0}^{1} (1-e^{-t'}) e^{-0.86\left(2.47m^2+4.4n^2\right)(1-t')} dt' \tag{7.1}$$

VIII. CONCLUSION

In both the problems, the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of semi-infinite rectangular beam have been investigated with the aid of integral transform techniques. The expressions are obtained in terms of Bessel's function in the form of infinite series. The results that are obtained can be applied to the design of useful structures or machines in engineering applications.

Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions

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