Thermal Stresses of Semi-Infinite Hollow Cylinder with Internal Heat Source

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Abstract- **This paper is concerned with the determination of temperature distribution, displacement function and thermal stresses of semi-infinite hollow cylinder occupying the space** $D=\{a \le r \le b,\ 0 \le z \le \infty\}$. **The governing heat conduction equation has been solved by using integral transform techniques. The results are obtained in terms of Bessel's function in the form of infinite series.**

Key Words: **Semi-infinite hollow cylinder, internal heat source, integral transform, transient problem.**

I. INTRODUCTION

In **2009, Kamdi, Khobragade** and **Durge [1]** studied transient thermoelastic problem for a circular solid cylinder with radiation. **Walde** and **Khobragade [2]** discussed transient thermoelastic problem of a finite length hollow cylinder. **Kulkarni and Khobragade [3]** derived thermal stresses of a finite length hollow cylinder. **Warbhe** and **Khobragade [4]** discussed numerical study of transient thermoelastic problem of a finite length hollow cylinder. **Lamba** and **Khobragade [5]** studied analysis of coupled thermal stresses in a axisymmetric hollow cylinder**. Hiranwar** and **Khobragade [6**] studied thermoelastic problem of a cylinder with internal heat sources. **Bagde** and **Khobragade [7**] discussed **heat** conduction problem for a finite elliptic cylinder**.**

Khobragade, Khalsa and Kulkarni [8] investigated thermal deflection of a finite length hollow cylinder due to heat generation. **Khobragade [9]** studied **thermoelastic** analysis of a thick hollow cylinder with radiation conditions**. Ghume, Mahakalkar and Khobragade [10]** derived interior thermo elastic solution of a hollow cylinder. **Chauthale, Singru and Khobragade [11]** studied thermal stress analysis of a thick hollow cylinder. **Singru and Khobragade [12]** developed integral transform methods for inverse problem of heat conduction with known boundary of semi-infinite hollow cylinder and its stresses.

In the present paper, an attempt is made to determine the temperature distribution, displacement function and thermal stresses of semi-infinite hollow cylinder occupying the space $D=\{a \le r \le b,\ 0 \le z \le \infty\}$. Finite **Marchi-Zgrablich transform and** Fourier sine transform techniques are used to solve the problem.

II. FORMULATION OF THE PROBLEM

Consider semi-infinite hollow cylinder. The displacement function $\phi(r, z, t)$ satisfying the differential equation as **Khobragade [9**] is

$$\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{\partial^2 \phi}{\partial z^2}=\left(\frac{1+\nu}{1-\nu}\right)a_t T \tag{2.1}$$

with $\phi = 0$ at $r=a$ and $r=b$ (2.2)

where ν and a_t are Poisson ratio and linear coefficient of thermal expansion of the material of the cylinder respectively and $T(r, z, t)$ is the heating temperature of the cylinder at time t satisfying the differential equation as **Khobragade [9]** is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]+\frac{g(r,z,t)}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.3}$$

where $\kappa = K/\rho c$ is the thermal diffusivity of the material of the cylinder, K is the conductivity of the medium, c is its specific heat and ρ is its calorific capacity (which is assumed to be constant) respectively.

subject to the initial and boundary conditions

$$M_t(T,1,0,0)=F \text{ for all } a \le r \le b,\ 0 \le z \le \infty \tag{2.4}$$

$$M_r(T,1,k_1,a)=F_1(z,t), \text{ for all } 0 \le z \le \infty, \tag{2.5}$$

$$M_r(T,1,k_2,b)=F_2(z,t) \text{ for all } 0 \le z \le \infty \tag{2.6}$$

$$M_z(T,1,0,0)=0 \text{ for all } a\le r\le b \tag{2.7}$$

$$M_z(T,1,0,\infty)=0 \text{ for all } a\le r\le b \tag{2.8}$$

being:

$$M_{\vartheta}(f,\bar{k},\bar{\bar{k}},\$)=(\bar{k}\ f+\bar{\bar{k}}\ \hat{f})_{\vartheta=\$}$$

where the prime (^) denotes differentiation with respect to ϑ, radiation constants are $\bar{k}$ and $\bar{\bar{k}}$ on the curved surfaces of the cylinder respectively.

The radial and axial displacement U and W satisfy the uncoupled thermoelastic equation as **Khobragade [9]** are

$$\nabla^2 U-\frac{U}{r^2}+(1-2\nu)^{-1}\frac{\partial e}{\partial r}=2\left(\frac{1+\nu}{1-2\nu}\right)a_t\frac{\partial T}{\partial r} \tag{2.9}$$

$$\nabla^2 W+(1-2\nu)^{-1}\frac{\partial e}{\partial z}=2\left(\frac{1+\nu}{1-2\nu}\right)a_t\frac{\partial T}{\partial z} \tag{2.10}$$

where

$$e=\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{\partial W}{\partial r} \tag{2.11}$$

$$U=\frac{\partial\phi}{\partial r}, \tag{2.12}$$

$$W=\frac{\partial\phi}{\partial z} \tag{2.13}$$

The stress functions are given by

$$\tau_{rz}(a,z,t)=0,\ \tau_{rz}(b,z,t)=0,\ \tau_{rz}(r,0,t)=0 \tag{2.14}$$

$$\sigma_r(a,z,t)=p_i,\ \sigma_r(b,z,t)=-p_o,\ \sigma_z(r,0,t)=0 \tag{2.15}$$

where p_i and p_o are the surface pressure assumed to be uniform over the boundaries of the cylinder.

The stress functions are expressed in terms of the displacement components by the following relations as **Khobragade [9]:**

$$\sigma_r=(\lambda+2G)\frac{\partial U}{\partial r}+\lambda\left(\frac{U}{r}+\frac{\partial W}{\partial z}\right) \tag{2.16}$$

$$\sigma_z=(\lambda+2G)\frac{\partial W}{\partial z}+\lambda\left(\frac{\partial U}{\partial r}+\frac{U}{r}\right) \tag{2.17}$$

$$\sigma_\theta=(\lambda+2G)\frac{U}{r}+\lambda\left(\frac{\partial U}{\partial r}+\frac{\partial W}{\partial z}\right) \tag{2.18}$$

$$\tau_{rz}=G\left(\frac{\partial W}{\partial r}+\frac{\partial U}{\partial z}\right) \tag{2.19}$$

where $\lambda=2G\nu/(1-2\nu)$ is Lame's constant, G is the shear modulus and U, W are the displacement components. Equations (2.1)-(2.19) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE OF THE PROBLEM

Applying Fourier sine transform and Marchi-Zgrablich transform to the equations (2.3), (2.4) and (2.6) one obtains

$$\bar{T}^*(n,m,t)=e^{-\alpha p^2 t}\left[\bar{F}^*+\int_0^t \Pi e^{\alpha p^2 t'}dt'\right] \tag{3.1}$$

where constants involved $\bar{T}^*(n,m,t)$ are obtained by using boundary conditions (2.4). Finally applying the inversion of Marchi-Zgrablich transform and Fourier sine transform one obtains the expressions of the temperature distribution $T(r,z,t)$ for heating processes as

$$T(r,z,t)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{C_n}\sum_{m=1}^{\infty}\sin(pz)S_0(k_1,k_2,\mu_n r)\times e^{-\alpha p^2 t}\left[\bar{F}^*+\int_0^t \Pi e^{\alpha p^2 t'}dt'\right] \tag{3.2}$$

where n is the transformation parameter as defined in appendix, m is the Fourier sine transform parameter and

$$C_n=\int_a^b [r S_0(\alpha,\beta,\mu_n r)]^2 dr\ .$$

IV. DETERMINATION OF DISPLACEMENT AND STRESS FUNCTION

Substituting the value of temperature distribution T(r,z,t) from (3.2) in equation (2.1) one obtains the thermo elastic displacement function $\phi(r,z,t)$ as

$$\phi(r, z, t) = \frac{r^2 a_t}{4\pi}\left(\frac{1+\nu}{1-\nu}\right)\sum_{n=1}^{\infty}\frac{1}{C_n}\sum_{m=1}^{\infty} \sin(pz)$$
$$\times S_0(k_1, k_2, \mu_n r)\Omega(t) \qquad (4.1)$$

Where $\Omega(t) = e^{-\alpha p^2 t}\left[\overline{F}^* + \int_0^t \Pi\, e^{\alpha p^2 t'} dt'\right]$

Using (4.1) in the equations (2.12) and (2.13) one obtains

$$U = \frac{r\, a_t}{4\pi}\left(\frac{1+\nu}{1-\nu}\right)\sum_{n=1}^{\infty}\frac{1}{C_n}\sum_{m=1}^{\infty}\Omega(t)$$
$$\times\left[2S_0(k_1, k_2, \mu_n r) + r\mu_n S'_0(k_1, k_2, \mu_n r)\right] \qquad (4.2)$$

$$W = \frac{r^2\, a_t}{4\pi}\left(\frac{1+\nu}{1-\nu}\right)\sum_{n=1}^{\infty}\frac{1}{C_n}\sum_{m=1}^{\infty}\Omega(t)\, p\cos(pz)$$
$$\times S_0(k_1, k_2, \mu_n r) \qquad (4.3)$$

Substitution the value of (4.2), (4.3) in (2.15) to (2.18) one obtains the stress functions as

$$\sigma_r = \frac{a_t}{4\pi\xi}\left(\frac{1+\nu}{1-\nu}\right)\sum_{m,n=1}^{\infty}\frac{\Omega(t)\sin pz}{C_n}$$

$$\begin{bmatrix}(\lambda+2G)\left(r^2\mu_n^{\,2} S''_0(k_1, k_2, \mu_n r)\right) + 4r\mu_n S'_0(k_1, k_2, \mu_n r) \\ + 2S_0(k_1, k_2, \mu_n r)\end{bmatrix}$$
$$\times\lambda\begin{bmatrix}(2-r^2p^2)S_0(k_1, k_2, \mu_n r) \\ + r\,\mu_n S'_0(k_1, k_2, \mu_n r)\end{bmatrix} \qquad (4.4)$$

$$\sigma_z = -\frac{a_t}{4\pi}\left(\frac{1+\nu}{1-\nu}\right)\sum_{m,n=1}^{\infty}\left(\frac{\Omega(t)\sin pz}{C_n}\right)$$
$$\times[(\lambda+2G)r^2p^2 S_0(k_1, k_2, \mu_n r)$$
$$-\lambda\begin{pmatrix}r^2\mu_n^{\,2} S''_0(k_1, k_2, \mu_n r) + 5r\mu_n S'_0(k_1, k_2, \mu_n r) \\ + 4S_0(k_1, k_2, \mu_n r)\end{pmatrix}]$$
$$(4.5)$$

$$\sigma_\theta = \frac{a_t}{4\pi}\left(\frac{1+\nu}{1-\nu}\right)\sum_{m,n=1}^{\infty}\left(\frac{\Omega(t)\sin pz}{C_n}\right)(\lambda+2G)$$
$$\times\left[r\mu_n S'_0(k_1, k_2, \mu_n r) + 2S_0(k_1, k_2, \mu_n r)\right]$$
$$+\lambda\begin{bmatrix}r^2\mu_n^{\,2} S''_0(k_1, k_2, \mu_n r) + 4r\mu_n S'_0(k_1, k_2, \mu_n r) \\ + (2 - r^2p^2)S_0(k_1, k_2, \mu_n r)\end{bmatrix}$$
$$(4.6)$$

$$\tau_{rz} = \frac{r a_t G}{2\pi}\left(\frac{1+\nu}{1-\nu}\right)\sum_{m,n=1}^{\infty}\left(\frac{\Omega(t)\, p\cos(p_m z)}{C_n}\right)$$
$$\times\left[r\mu_n S'_0(k_1, k_2, \mu_n r) + 2S_0(k_1, k_2, \mu_n r)\right] \qquad (4.7)$$

V. SPECIAL CASE

Set $F(z, t) = \dfrac{z(e^{-t})}{1+z^2}\delta(r - r_0)$ (5.1)

Applying finite Marchi Zgrablich transform and Fourier sine transform to the equation (5.1) one obtains

$$\overline{F}^*(n, m, t) = (\pi/2)(e^{-(p+t)})\, r_0\, S_0(k_1, k_2, \mu_n r_0)$$
$$(5.2)$$

Substituting the value of (5.2) in the equations (3.2) one obtains

$$T(r, z, t) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{C_n}\sum_{m=1}^{\infty} \sin(pz)\, S_0(k_1, k_2, \mu_n r)$$
$$\times\begin{bmatrix}(\pi/2)(e^{-(\alpha p^2 t + p + t)})\, r_0\, S_0(k_1, k_2, \mu_n r_0) \\ + \int_0^t \Pi\, e^{-\alpha p^2(t-t')} dt'\end{bmatrix}$$
$$(5.3)$$

Where $p = m\pi$.

VI. NUMERICAL RESULTS

Set $a = 2m,\ b = 2.5m,$ t = 1 sec **,** $k_1 = 0.25 = k_2$

Substituting these values in equations (5.3) we get

$$T(r, z, t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\sum_{m=1}^{\infty} \sin(pz)\, S_0(0.25, 0.25, \mu_n r)$$

$$\times \left[\begin{array}{l} (0.5)(e^{-(\alpha p^2+p+1)})\, r_0\, S_0(0.25,\, 0.25,\, \mu_n r_0) \\ + \dfrac{1}{\pi}\displaystyle\int_0^1 \Pi\, e^{-\alpha p^2(1-t')} dt' \end{array} \right] \tag{6.1}$$

VII. MECHANICAL AND THERMAL PROPERTIES

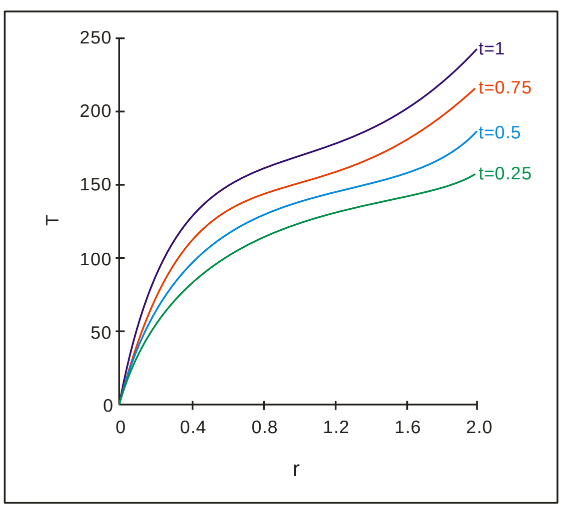
To interpret the numerical computation we consider material properties of low carbon steel (AISI 1119), which can be used for medium duty shafts, studs, pins, distributor cams, cam shafts, and universal joints having mechanical and thermal properties:

$\kappa = 13.97[\mu m/s^2]$ $\upsilon = 0.29,$

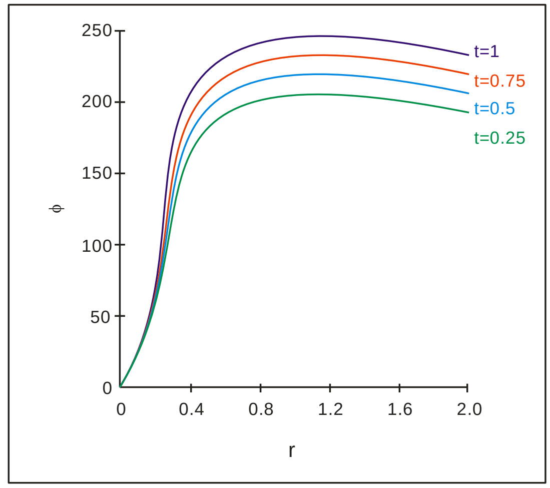
$\lambda = 51.9\,[W/(m-K)]$ and $a_t = 14.7\,\mu m/m-{}^0C$.

VIII. CONCLUSION

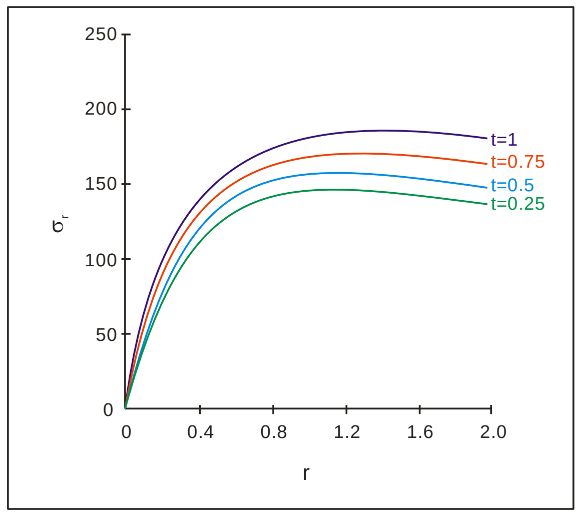
In this paper, the temperature distributions, displacement function and thermal stress of semi-infinite hollow cylinder $a \le r \le b,\ \ 0 \le z \le \infty$ have been determined for internal heat source, with the known boundary conditions. We develop the analysis for the temperature field by introducing the transformation defined by Zgrablich et al. **[118]** and Fourier sine transform for radiations type boundary conditions. The series solutions converge provided we take sufficient number of terms in the series. Since the thickness of cylinder is very small, the series solution given here will be definitely convergent. Any particular case can be derived by assigning suitable values to the parameters and functions in the series expressions. The results that are obtained can be applied to the design of useful structures or machines in engineering applications.



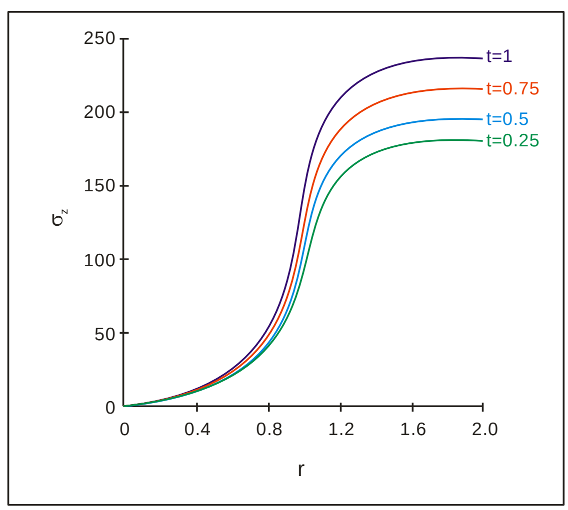
Graph 1: Temperature distribution versus r



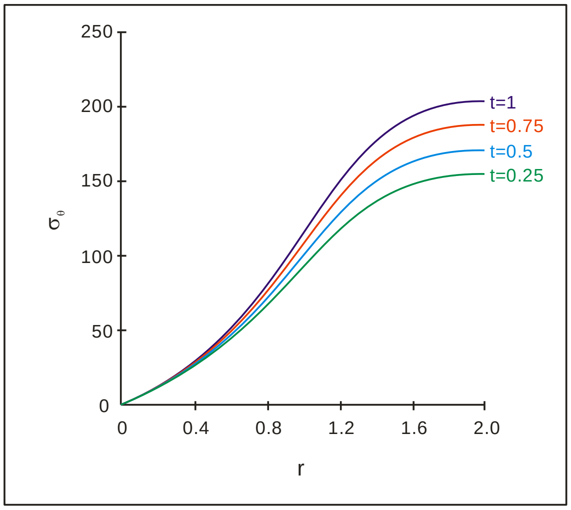
Graph 2: Displacement function versus r



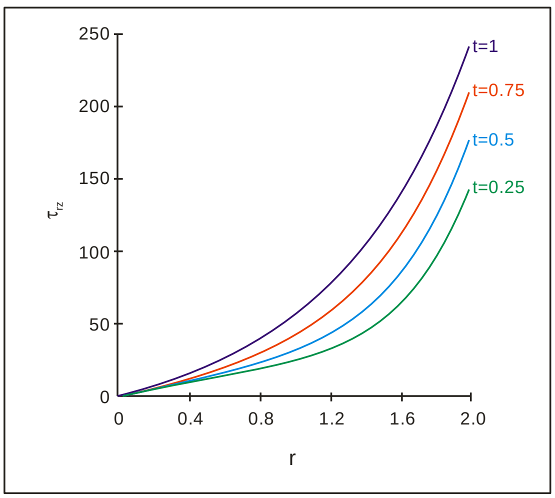
Graph 3: Stress Function versus r



Graph 4: Stress Function versus r



Graph 5: Stress Function versus r



Graph 6: Stress Function versus r

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APPENDIX

FINITE MARCHI-ZGRABLICH INTEGRAL TRANSFORM

The finite Marchi-Zgrablich integral transform of $f(r)$ is defined as

$$\bar{f}_p(m) = \int_a^b r\, f(r)\, S_p(\alpha, \beta, \mu_m r)\, dr \qquad (1)$$

where α_1, α_2, β_1 and β_2 are the constants involved in the boundary conditions $\alpha_1 f(r) + \alpha_2 \; f'(r)\big|_{r=a} = 0$ and $\beta_1 f(r) + \beta_2 \; f'(r)\big|_{r=b} = 0$ for the differential equation $f''(r) + (1/r) f'(r) - (p^2 / r^2) f(r) = 0$, $\bar{f}_p(n)$ is the transform of $f(r)$ with respect to kernel $S_p(\alpha, \beta, \mu_m r)$ and weight function r

The inversion of equation (1) is given by

$$f(r) = \sum_{m=1}^{\infty} \frac{\bar{f}_p(m)\, S_p(\alpha, \beta, \mu_m r)}{\int_a^b [r\, S_p(\alpha, \beta, \mu_m r)]^2\, dr} \qquad (2)$$

where kernel function $S_p(\alpha, \beta, \mu_m r)$ can be defined as

$$S_p(\alpha, \beta, \mu_m r) = J_p(\mu_m r)[Y_p(\alpha, \mu_m a) + Y_p(\beta, \mu_m b)]$$
$$- Y_p(\mu_m r)[J_p(\alpha, \mu_m a) + J_p(\beta, \mu_m b)] \qquad (3)$$

and $J_p(\mu r)$ and $Y_p(\mu r)$ are Bessel function of first and second kind respectively.

OPERATIONAL PROPERTY:

$$\int_a^b r^2 \left(f''(r) + (1/r) f'(r) - (p^2/r^2) f(r) \right) S_p(\alpha, \beta, \mu_m r)\, dr$$

$$= (b/\beta_2) S_p(\alpha, \beta, \mu_m b) \left[\beta_1 f(r) + \beta_2 f'(r) \right]_{r=b}$$

$$- (a/\alpha_2) S_p(\alpha, \beta, \mu_m a) \left[\alpha_1 f(r) + \alpha_2 f'(r) \right]_{r=a} - \mu_m^2 \bar{f}_p(m) \quad (4)$$