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Inverse Thermoelastic Problem of Semi-Infinite Annular Beam

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Abstract- **This paper is concerned with inverse thermoelastic problem in which we need to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of a semi-infinite annular beam when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.**

Key Words**: Semi-infinite annular beam, thermoelastic problem, Integral transform, inverse problem**

I. INTRODUCTION

In 1968, Ozisik [1] has published a book on Boundary Value Problem of Heat Conduction. **Varghese** and **Khobragade [2]** studied inverse transient thermoelastic problem of a thin annular disc. **Varghese** and **Khobragade** [3] discussed transient thermoelastic analysis in thin annular disc with radiation type boundaries conditions. **Hiranwar** and **Khobragade** [4] studied thermoelastic problem of a thin annular disc due to radiation. **Lamba** and **Khobragade** [5] derived thermal stresses of a thin annular disc due to partially distributed heat supply. **Walde** and **Khobragade [6]** discussed inverse thermoelastic problem of a thin annular disc due to heat generation. Further **Walde** and **Khobragade** [7] derived thermal deflection of a clamped annular disc due to heat generation.

Sutar and **Khobragade** [8] derived solution of an inverse thermoelastic problem of heat conduction with internal heat generation in an annular disc. **Khobragade [9]** has developed thermoelastic analysis of a thick annular disc with radiation conditions**. Further Khobragade [10]** derived thermal deflection of an annular disc due to heat generation. **Navlekar, Warbhe and Khobragade [11]** studied heat transfer and thermal stresses of a thick annular disc due to heat generation. **Ovais Ahmed, Khobragade and Khalsa [12]** investigated optimum thermal stresses of a thick annular disc due to partially distributed heat supply. **Singru and Khobragade [13]** developed integral transform methods for inverse problem of heat conduction with known boundary of semi-infinite hollow cylinder and its stresses . **Pakade and Khobragade [14]** studied transient thermoelastic problem of Semi-Infinite circular beam with internal heat sources.

In this paper, an attempt has been made to solve two inverse problems of thermoelasticity.

In the first steady-state problem, an attempt has been made to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses on the outer curved surface of annular beam. The temperature is maintained at zero on the plane surfaces of a beam and at inner and outer curved surfaces, it is maintained at u(z) and f(z) respectively.

In the second transient problem, an attempt has been made to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stress functions on the outer curved surface of annular beam. The temperature is maintained at zero on the plane surfaces of a beam and at inner and outer curved surfaces, it is maintained at u(z,t) and f(z,t) respectively.

II. STATEMENT OF THE PROBLEM-I

Consider semi-infinite annular beam occupying the space D : $a \le r \le b,\ 0 \le z \le \infty$. The differential equation governing the displacement function $U(r,z)$ as [1] is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)a_t T \qquad (2.1)$$

with $U = 0$ at $r = a$ and $r = b$ (2.2)

where ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the beam respectively and $T(r,z)$ is the temperature of the beam satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (2.3)$$

subject to the boundary conditions

$$T(a,z) = u(z) \qquad (2.4)$$

$$T(b,z) = g(z) \text{ (unknown)} \qquad (2.5)$$

$$T(r,z)\Big|_{z=0} = 0 \tag{2.6}$$

$$T(r,z)\Big|_{z=\infty} = 0 \tag{2.7}$$

The interior condition

$$T(\xi,z) = f(z) \quad , \quad 0 < \xi < a \quad \text{(known)} \tag{2.8}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial U}{\partial r} \tag{2.9}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \tag{2.10}$$

where μ is the Lame's constant, while each of the stress function σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the beam in the plane state of stress. Equations (2.1) to (2.10) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying Fourier sine transform to the equations (2.3), (2.4), (2.5) and (2.8) and using (2.6), (2.7) one obtains

$$\frac{d^2\bar{T}_S}{dr^2} + \frac{1}{r}\frac{d\bar{T}_S}{dr} - p^2\bar{T}_S = 0 \tag{3.1}$$

where $p^2 = m^2\pi^2$ (3.2)

$$\bar{T}_S(a,m) = \bar{u}_S(m) \tag{3.3}$$

$$\bar{T}_S(b,m) = \bar{g}_S(m) \tag{3.4}$$

$$\bar{T}_S(\xi,m) = \bar{f}_S(m) \tag{3.5}$$

where $\bar{T}_S$ denotes Fourier sine transform of T and m is sine transform parameter.

Equation (3.1) is a Bessel's equation whose solution gives

$$\bar{T}_s(r,m) = AI_0(pr) + BK_0(pr) \tag{3.6}$$

where A, B are constants and $I_0(pr)$, $K_0(pr)$ are modified Bessel's functions of first and second kind of order zero respectively.

Applying the boundary conditions (3.3) and (3.5) to the equation (3.6) one obtains

$$AI_0(pa) + BK_0(pa) = \bar{u}_s(m) \tag{3.7}$$

$$AI_0(p\xi) + BK_0(p\xi) = \bar{f}_s(m) \tag{3.8}$$

Solving (3.7) and (3.8) one obtains

$$A = \frac{\bar{f}_s(m)K_0(pa)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)} - \frac{\bar{u}_s(m)K_0(p\xi)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}$$

$$B = -\frac{\bar{f}_s(m)I_0(pa)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)} + \frac{\bar{u}_s(m)I_0(p\xi)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}$$

Substituting the values of A and B in (3.6) one obtains

$$\bar{T}_s(r,m) = \bar{f}_s(m)\left[\frac{I_0(pr)K_0(pa) - K_0(pr)I_0(pa)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}\right] - \bar{u}_s(m)\left[\frac{I_0(pr)K_0(p\xi) - K_0(pr)I_0(p\xi)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}\right] \tag{3.9}$$

Using the condition (3.4) to the solution (3.9) one obtains

$$\bar{g}_s(m) = \bar{f}_s(m)\left[\frac{I_0(pb)K_0(pa) - K_0(pb)I_0(pa)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}\right] - \bar{u}_s(m)\left[\frac{I_0(pb)K_0(p\xi) - K_0(pb)I_0(p\xi)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}\right] \tag{3.10}$$

Applying inverse Fourier sine transform to the equations (3.9) and (3.10) one obtain the expression for the temperature distribution T(r,z) and the unknown temperature gradient g(z) as

$$T(r,z) = \frac{1}{\pi}\sum_{m=1}^{\infty}\bar{f}_s(m)\sin pz\left[\frac{I_0(pr)K_0(pa) - K_0(pr)I_0(pa)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}\right] - \frac{1}{\pi}\sum_{m=1}^{\infty}\bar{u}_s(m)\sin pz\left[\frac{I_0(pr)K_0(p\xi) - K_0(pr)I_0(p\xi)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}\right] \tag{3.11}$$

$$g(z) = \frac{1}{\pi}\sum_{m=1}^{\infty}\bar{f}_s(m)\sin pz\left[\frac{I_0(pb)K_0(pa) - K_0(pb)I_0(pa)}{I_0(p\xi)K_0(pa) - K_0(p\xi)I_0(pa)}\right]$$

$$-\frac{1}{\pi}\sum_{m=1}^{\infty}\bar{u}_s(m)\sin pz\left[\frac{I_0(pb)K_0(p\xi)-K_0(pb)I_0(p\xi)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right] \tag{3.12}$$

where $\bar{f}_s(m)=\int_0^{\infty} f(z)\sin pz\,dz$ and

$$\bar{u}_s(m)=\int_0^{\infty} u(z)\sin pz\,dz$$

IV. DETERMINATION OF THERMOELASTIC DISPLACEMENT FUNCTION

Substituting the value of T(r,z) from (3.11) in (2.1) one obtains the thermoelastic displacement function U(r,z) as

$$U(r,z)=-(1+\nu)a_t\left[\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\bar{f}_s(m)}{p^2}\sin pz\right]\times\left[\frac{I_0(pr)K_0(pa)-K_0(pr)I_0(pa)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right]$$
$$+(1+\nu)a_t\left[\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\bar{u}_s(m)}{p^2}\sin pz\right]\times\left[\frac{I_0(pr)K_0(p\xi)-K_0(pr)I_0(p\xi)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right] \tag{4.1}$$

V. DETERMINATION OF STRESS FUNCTIONS

Using the equation (4.1) in (2.9) and (2.10) the stress functions are obtained as

$$\sigma_{rr}=\frac{2\mu}{r}(1+\nu)a_t\left[\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\bar{f}_s(m)}{p}\sin pz\right]\times\left[\frac{I_0'(pr)K_0(pa)-K_0'(pr)I_0(pa)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right]$$
$$-\frac{2\mu}{r}(1+\nu)a_t\left[\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\bar{u}(m)}{p}\sin pz\right]\times\left[\frac{I_0'(pr)K_0(p\xi)-K_0'(pr)I_0(p\xi)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right] \tag{5.1}$$

$$\sigma_{\theta\theta}=2\mu(1+\nu)a_t\left[\frac{1}{\pi}\sum_{m=1}^{\infty}\bar{f}_s(m)\sin pz\right]\times\left[\frac{I_0''(pr)K_0(pa)-K_0''(pr)I_0(pa)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right]$$
$$-2\mu(1+\nu)a_t\left[\frac{1}{\pi}\sum_{m=1}^{\infty}\bar{u}_s(m)\sin pz\right]\times\left[\frac{I_0''(pr)K_0(p\xi)-K_0''(pr)I_0(p\xi)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right] \tag{5.2}$$

VI. SPECIAL CASE

Set $f(z)=\dfrac{z\xi}{1+z^2}$, $u(z)=\dfrac{za}{1+z^2}$ (6.1)

Applying Fourier sine transform to the equation (6.1) one obtains

$$\bar{f}_s(m)=\int_0^{\infty}\left(\frac{z\xi}{1+z^2}\right)\sin(pz)dz$$
$$=\left(\frac{\pi\xi}{2}\right)\left[e^{-p}\right] \tag{6.2}$$

$$\bar{u}_s(m)=\int_0^{\infty}\left(\frac{za}{1+z^2}\right)\sin(pz)dz$$
$$=\left(\frac{\pi a}{2}\right)\left[e^{-p}\right] \tag{6.3}$$

Substituting the values of $\bar{f}_s(m)$ and $\bar{u}_s(m)$ from (6.2) and (6.3) in the equations (3.11), (3.12) one obtains

$$T(r,z)=\frac{\xi}{2}\sum_{m=1}^{\infty}\sin pz\left[\frac{I_0(pr)K_0(pa)-K_0(pr)I_0(pa)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right]$$
$$-\frac{a}{2}\sum_{m=1}^{\infty}\sin pz\left[\frac{I_0(pr)K_0(p\xi)-K_0(pr)I_0(p\xi)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right] \tag{6.4}$$

$$g(z)=\frac{\xi}{2}\sum_{m=1}^{\infty}\sin pz\left[\frac{I_0(pb)K_0(pa)-K_0(pb)I_0(pa)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right]$$

$$-\frac{a}{2}\sum_{m=1}^{\infty} \sin pz\left[\frac{I_0(pb)K_0(p\xi)-K_0(pb)I_0(p\xi)}{I_0(p\xi)K_0(pa)-K_0(p\xi)I_0(pa)}\right] \tag{6.5}$$

VII. NUMERICAL RESULTS

Set $\alpha=\frac{1}{2}$, π =3.14, a = 0.5 m, b=2 m, $\xi=1.5$ m and in the equation (6.4) to obtain

$$\frac{g(z)}{\alpha}=\sum_{m=1}^{\infty} \sin(3.14mz)$$
$$\times\left\{\left[\frac{I_0(6.28m)K_0(3.14m)-K_0(6.28m)I_0(3.14m)}{I_0(4.71m)K_0(3.14m)-K_0(4.71m)I_0(3.14m)}\right](1.5)\right.$$
$$\left.-\left[\frac{I_0(6.28m)K_0(4.71m)-K_0(6.28m)I_0(4.71m)}{I_0(4.71m)K_0(3.14m)-K_0(4.71m)I_0(3.14m)}\right](0.5)\right\} \tag{7.1}$$

VIII. STATEMENT OF THE PROBLEM-II

Consider semi-infinite annular beam occupying the space D : $a \le r \le b,\ 0 \le z \le \infty$. The differential equation governing the displacement function U(r,z,t) as [1] is

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=(1+\nu)a_t T \tag{8.1}$$

with U = 0 at $r = a$ and $r = b$ (8.2)

where ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the beam respectively and T(r,z,t) is the temperature of the beam satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{k}\frac{\partial T}{\partial t} \tag{8.3}$$

subject to the initial condition

$$T(r,z,0)=0 \tag{8.4}$$

The boundary conditions are

$$T(a,z,t)=u(z,t) \tag{8.5}$$

$$T(b,z,t)=g(z,t) \text{ (unknown)} \tag{8.6}$$

$$T(r,z,t)\big|_{z=0}=0 \tag{8.7}$$

$$T(r,z,t)\big|_{z=\infty}=0 \tag{8.8}$$

The interior condition is

$$T(\xi,z,t)=f(z,t),\ a<\xi<b \text{ (known)} \tag{8.9}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr}=-2\mu\frac{1}{r}\frac{\partial U}{\partial r} \tag{8.10}$$

$$\sigma_{\theta\theta}=-2\mu\frac{\partial^2 U}{\partial r^2} \tag{8.11}$$

where μ is the Lame's constant, while each of the stress function σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the beam in the plane state of stress.

Equations (8.1) to (8.11) constitute the mathematical formulation of the problem under consideration.

IX. SOLUTION OF THE PROBLEM

Applying Fourier sine transform to the equations (8.3), (8.4), (8.5), (8.6) and (8.9) and using (8.7), (8.8) one obtains

$$\frac{d^2\bar{T}_S}{dr^2}+\frac{1}{r}\frac{d\bar{T}_S}{dr}-p^2\bar{T}_S=\frac{1}{k}\frac{d\bar{T}_S}{dt} \tag{9.1}$$

where $p^2=m^2\pi^2$

$$\bar{T}_S(r,m,0)=0 \tag{9.2}$$

$$\bar{T}_S(a,m,t)=\bar{u}_S(m,t) \tag{9.3}$$

$$\bar{T}_S(b,m,t)=\bar{g}_S(m,t) \tag{9.4}$$

$$\bar{T}_S(\xi,m,t)=\bar{f}_S(m,t) \tag{9.5}$$

where $\bar{T}_S$ denotes Fourier sine transform of T and m is sine transform parameter.

Applying Laplace transform to the equations (9.1), (9.3), (9.4), (9.5) and using the conditions (9.2) one obtains

$$\frac{d^2\bar{T}_S^*}{dr^2}+\frac{1}{r}\frac{d\bar{T}_S^*}{dr}-q^2\bar{T}_S^*=0 \tag{9.6}$$

where $q^2=p^2+\frac{s}{k}$ (9.7)

$$\bar{T}_S^*(a,m,t)=\bar{u}_S^*(m,t) \tag{9.8}$$

$$\bar{T}_S^*(b,m,t)=\bar{g}_S^*(m,t) \tag{9.9}$$

$$\bar{T}_s^*(\xi,m,t)=\bar{f}_s^*(m,t) \tag{9.10}$$

where $\bar{T}_s^*$ denotes Laplace transform of $\bar{T}_s$ and s is Laplace transform parameter.

Equation (9.6) is a Bessel's equation whose solution gives

$$\bar{T}_s^*(r,m,s)=AI_0(qr)+BK_0(qr) \tag{9.11}$$

where A, B are constants and I_0 , K_0 are modified Bessel's functions of first and second kind of order zero respectively.

Using (9.8) and (9.10) in (9.11) one obtains

$$AI_0(qa)+BK_0(qa)=\bar{u}_s(m,s) \tag{9.12}$$

$$AI_0(q\xi)+BK_0(q\xi)=\bar{f}_s(m,s) \tag{9.13}$$

Solving (7.9.12) and (7.9.13) one obtains

$$A=\frac{\bar{f}_s^*(m,s)K_0(qa)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}-\frac{\bar{u}_s^*(m,s)K_0(q\xi)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}$$

$$B=-\frac{\bar{f}_s^*(m,s)I_0(qa)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}+\frac{\bar{u}_s^*(m,s)I_0(q\xi)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}$$

Substituting the values of A and B in (9.11) and using condition (8.9) one obtains

$$\bar{T}_s^*(r,m,s)=\bar{f}_s^*(m,s)\left[\frac{I_0(qr)K_0(qa)-K_0(qr)I_0(qa)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}\right]-\bar{u}_s^*(m,s)\left[\frac{I_0(qr)K_0(q\xi)-K_0(qr)I_0(q\xi)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}\right] \tag{9.14}$$

$$\bar{g}_s^*(m,s)=\bar{f}_s^*(m,s)\left[\frac{I_0(qb)K_0(qa)-K_0(qb)I_0(qa)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}\right]-\bar{u}_s^*(m,s)\left[\frac{I_0(qb)K_0(q\xi)-K_0(qb)I_0(q\xi)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}\right] \tag{9.15}$$

Applying inverse Laplace transform) to the equation (9.14) one obtains

$$\bar{T}_s(r,m,t)=L^{-1}\left\{\bar{f}_s^*(m,s)\times\bar{g}_1^*(s)\right\}-L^{-1}\left\{\bar{u}_s^*(m,s)\times\bar{g}_2^*(s)\right\} \tag{9.16}$$

where

$$\bar{g}_1^*(s)=\left[\frac{I_0(qr)K_0(qa)-K_0(qr)I_0(qa)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}\right] \tag{9.17}$$

$$\bar{g}_2^*(s)=\left[\frac{I_0(qr)K_0(q\xi)-K_0(qr)I_0(q\xi)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}\right] \tag{9.18}$$

<u>To calculate the inverse Laplace transform of (9.17):</u>

Applying inverse Laplace transform to the equation (9.17) one obtains

$$\bar{g}_1(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}\left[\frac{I_0(qr)K_0(qa)-K_0(qr)I_0(qa)}{I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)}\right]ds \tag{9.19}$$

where c is greater than the real part of the singularities of the integer and the integrand is a single valued function of s. The poles of the integrand are at the points

$$s=s_n=-k\left[p^2+\lambda_n^2\right]$$

where λ_n are the positive roots of the transcendental equation

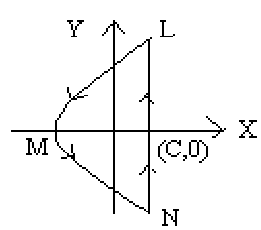
$$J_0(\lambda_n r)Y_0(\lambda_n a)-Y_0(\lambda_n r)J_0(\lambda_n a)=0 \tag{9.20}$$

The zeros of $I_0(q\xi)K_0(qa)-K_0(q\xi)I_0(qa)$ all are real and simple.

The poles of the integrand (9.18) are at

$$s=-k\left[p^2+\lambda_n^2\right] \qquad n = 1,2,3,\ldots$$

Using the contour of figure given below,



The integrand (9.19) is equal to $2\pi i$ times the sum of the residues at the poles of the integrand.

To find the residue at the point, one requires the result:

$$\left\{s\frac{d}{ds}(I_0(qa)K_0(q\xi)-K_0(qa)I_0(q\xi))\right\}$$
$$=\frac{a}{2kq}\left[(I_0{}'(qa)K_0(q\xi)-K_0{}'(qa)I_0(q\xi))\right]$$
$$+\frac{\xi}{2kq}\left[(I_0(qa)K_0{}'(q\xi)-K_0(qa)I_0{}'(q\xi))\right] \tag{9.21}$$

where $s=-k\left[p^2+\lambda_n{}^2\right]$, $q^2=p^2+\dfrac{s}{k}$ and

$$\frac{I_0(qa)}{I_0(q\xi)}=\frac{K_0(qa)}{K_0(q\xi)}=\frac{J_0(\lambda_n a)}{J_0(\lambda_n\xi)}$$

The equation (9.21) reduces to the form

$$(p^2+\lambda_n{}^2)\left[\frac{J_0{}^2(\lambda_n\xi)-J_0{}^2(\lambda_n a)}{2\lambda_n{}^2J_0(\lambda_n\xi)J_0(\lambda_n\xi)}\right] \tag{9.22}$$

Also

$$\left(I_0(qa)K_0(qr)-K_0(qa)I_0(qr)\right)\Big|_{s=-k(\lambda_n{}^2+p^2)}=-\frac{\pi}{2}$$
$$\times\left(J_0(\lambda_n a)Y_0(\lambda_n r)-Y_0(\lambda_n a)J_0(\lambda_n r)\right) \tag{9.23}$$

Using (9.21) and (9.23) in (9.19) one obtains

$$\bar{g}_1(t)=-\pi\lambda_n{}^2\frac{\left(J_0(\lambda_n a)Y_0(\lambda_n r)-Y_0(\lambda_n a)J_0(\lambda_n r)\right)}{\left(\lambda_n{}^2+p^2\right)\left(J_0{}^2(\lambda_n\xi)-J_0{}^2(\lambda_n a)\right)}$$
$$\times J_0(\lambda_n a)J_0(\lambda_n\xi)e^{-k(\lambda_n{}^2+p^2)t}$$

Applying the same arguments stated above to $\bar{g}_2{}^*(s)$, one obtains

$$\bar{g}_2(t)=-\pi\lambda_n{}^2\frac{\left(J_0(\lambda_n\xi)Y_0(\lambda_n r)-Y_0(\lambda_n\xi)J_0(\lambda_n r)\right)}{\left(\lambda_n{}^2+p^2\right)\left(J_0{}^2(\lambda_n\xi)-J_0{}^2(\lambda_n a)\right)}$$
$$\times J_0{}^2(\lambda_n\xi)e^{-k(\lambda_n{}^2+p^2)t}$$

Applying convolution theorem to the equation (9.16) one obtains

$$\bar{T}_S(r,m,s)=\pi\sum_{n=1}^{\infty}\frac{\lambda_n{}^2J_0(\lambda_n a)J_0(\lambda_n\xi)}{\left(\lambda_n{}^2+p^2\right)\left(J_0{}^2(\lambda_n a)-J_0{}^2(\lambda_n\xi)\right)}$$
$$\times\left[J_0(\lambda_n a)Y_0(\lambda_n r)-Y_0(\lambda_n a)J_0(\lambda_n r)\right]$$
$$\times\int_0^t\bar{f}_s(m,t')e^{-k(\lambda_n{}^2+p^2)(t-t')}dt'$$
$$-\pi\sum_{n=1}^{\infty}\frac{\lambda_n{}^2J_0{}^2(\lambda_n\xi)}{\left(\lambda_n{}^2+p^2\right)\left(J_0{}^2(\lambda_n a)-J_0{}^2(\lambda_n\xi)\right)}$$
$$\times\left[J_0(\lambda_n\xi)Y_0(\lambda_n r)-Y_0(\lambda_n\xi)J_0(\lambda_n r)\right]$$
$$\times\int_0^t\bar{u}_s(m,t')e^{-k(\lambda_n{}^2+p^2)(t-t')}dt' \tag{9.24}$$

Applying inverse Fourier sine transform to the equation (9.24) one obtains

$$T(r,z,t)=\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n{}^2J_0(\lambda_n a)J_0(\lambda_n\xi)}{\lambda_n{}^2+p^2}$$
$$\times\left(\frac{J_0(\lambda_n a)Y_0(\lambda_n r)-J_0(\lambda_n r)Y_0(\lambda_n a)}{J_0{}^2(\lambda_n a)-J_0{}^2(\lambda_n\xi)}\right)$$
$$\times\int_0^t\bar{f}_s(m,t')e^{-k(\lambda_n{}^2+p^2)(t-t')}dt'$$
$$-\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n{}^2J_0{}^2(\lambda_n\xi)}{\lambda_n{}^2+p^2}$$
$$\times\left(\frac{J_0(\lambda_n\xi)Y_0(\lambda_n r)-J_0(\lambda_n r)Y_0(\lambda_n\xi)}{J_0{}^2(\lambda_n a)-J_0{}^2(\lambda_n\xi)}\right)$$
$$\times\int_0^t\bar{u}_s(m,t')e^{-k(\lambda_n{}^2+p^2)(t-t')}dt' \tag{9.25}$$

where m, n are positive integers.

Applying inverse Laplace transform and then inverse Fourier sine transform to the equation (9.15) one obtains the expression for unknown temperature gradient g(z,t) as

$$g(z,t)=\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n{}^2J_0(\lambda_n a)J_0(\lambda_n\xi)}{\lambda_n{}^2+p^2}$$
$$\times\left(\frac{J_0(\lambda_n a)Y_0(\lambda_n b)-J_0(\lambda_n b)Y_0(\lambda_n a)}{J_0{}^2(\lambda_n a)-J_0{}^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt'$$

$$-\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0^2(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n\xi)Y_0(\lambda_n b)-J_0(\lambda_n b)Y_0(\lambda_n\xi)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \overline{u}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt' \qquad (9.26)$$

where m, n are positive integers and λ_n are the positive roots of the transcendental equation

$$J_0(\lambda_n a)Y_0(\lambda_n b)-J_0(\lambda_n b)Y_0(\lambda_n a)=0 \qquad (9.27)$$

$$\overline{f}_s(m,t)=\int_0^\infty f(z,t)\sin pz\,dz \qquad \text{and}$$

$$\overline{u}_s(m,t)=\int_0^\infty u(z,t)\sin pz\,dz$$

X. DETERMINATION OF THERMOELASTIC DISPLACEMENT

Substituting the value of T(r,z,t) from (9.25) in (8.1) one obtains the thermoelastic displacement function U(r,z,t) as

$$U(r,z,t)=-(1+\nu)a_t\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{J_0(\lambda_n a)J_0(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n a)Y_0(\lambda_n r)-J_0(\lambda_n r)Y_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt'$$

$$+(1+\nu)a_t\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{J_0^2(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n\xi)Y_0(\lambda_n r)-J_0(\lambda_n r)Y_0(\lambda_n\xi)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \overline{u}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt' \qquad (10.1)$$

XI. DETERMINATION OF STRESS FUNCTIONS

Using (10.1) in (8.10) and (8.11) the stress functions are obtained as

$$\sigma_{rr}=\frac{2\mu}{r}(1+\nu)a_t\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n J_0(\lambda_n a)J_0(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n a)Y_0'(\lambda_n r)-J_0'(\lambda_n r)Y_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt'$$

$$-\frac{2\mu}{r}(1+\nu)a_t\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n J_0^2(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n\xi)Y_0'(\lambda_n r)-J_0'(\lambda_n r)Y_0(\lambda_n\xi)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \overline{u}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt' \qquad (11.1)$$

$$\sigma_{\theta\theta}=2\mu(1+\nu)a_t\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0(\lambda_n a)J_0(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n a)Y_0''(\lambda_n r)-J_0''(\lambda_n r)Y_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \overline{f}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt'$$

$$-2\mu(1+\nu)a_t\sum_{m=1}^{\infty}\sin pz\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0^2(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n\xi)Y_0''(\lambda_n r)-J_0''(\lambda_n r)Y_0(\lambda_n\xi)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \bar{u}_s(m,t')e^{-k(\lambda_n^2+p^2)(t-t')}dt' \qquad (11.2)$$

XII. SPECIAL CASE

Set $f(z,t)=\left(\frac{z\xi}{1+z^2}\right)(1-e^{-t})$,

$$u(z,t)=\left(\frac{za}{1+z^2}\right)(1-e^{-t}) \qquad (12.1)$$

Applying Fourier sine transform to the equation (12.1) one obtains

$$\bar{f}_s(m,t)=\int_0^\infty (1-e^{-t})\left(\frac{z\xi}{1+z^2}\right)dz$$

$$=(1-e^{-t})\left[\frac{\pi\xi}{2}e^{-p}\right] \qquad (12.2)$$

and

$$\bar{u}_s(m,t)=\int_0^\infty (1-e^{-t})\left(\frac{za}{1+z^2}\right)dz$$

$$=(1-e^{-t})\left[\frac{\pi a}{2}e^{-p}\right] \qquad (12.3)$$

where m is positive integer.

Substituting the values of $\bar{f}_s(m,t)$ and $\bar{u}_s(m,t)$ from (12.2) and (12.3) in the equations (9.25), (9.26) one obtains

$$T(r,z,t)=\left(\frac{\pi\xi}{2}\right)\sum_{m=1}^\infty e^{-p}\sin pz\sum_{n=1}^\infty \frac{\lambda_n^2 J_0(\lambda_n a)J_0(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n a)Y_0(\lambda_n r)-J_0(\lambda_n r)Y_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \left(1-e^{-t'}\right)e^{-k(\lambda_n^2+p^2)(t-t')}dt'$$

$$-\left(\frac{\pi a}{2}\right)\sum_{m=1}^\infty e^{-p}\sin(pz)\sum_{n=1}^\infty \frac{\lambda_n^2 J_0^2(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\left(\frac{J_0(\lambda_n\xi)Y_0(\lambda_n r)-J_0(\lambda_n r)Y_0(\lambda_n\xi)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \left(1-e^{-t'}\right)e^{-k(\lambda_n^2+p^2)(t-t')}dt' \qquad (12.4)$$

$$g(z,t)=\left(\frac{\pi\xi}{2}\right)\sum_{m=1}^\infty e^{-p}\sin pz\sum_{n=1}^\infty \frac{\lambda_n^2 J_0(\lambda_n a)J_0(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n a)Y_0(\lambda_n b)-J_0(\lambda_n b)Y_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \left(1-e^{-t'}\right)e^{-k(\lambda_n^2+p^2)(t-t')}dt'$$

$$-\left(\frac{\pi a}{2}\right)\sum_{m=1}^\infty e^{-p}\sin(pz)\sum_{n=1}^\infty \frac{\lambda_n^2 J_0^2(\lambda_n\xi)}{\lambda_n^2+p^2}$$

$$\times\left(\frac{J_0(\lambda_n\xi)Y_0(\lambda_n b)-J_0(\lambda_n b)Y_0(\lambda_n\xi)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n\xi)}\right)$$

$$\times\int_0^t \left(1-e^{-t'}\right)e^{-k(\lambda_n^2+p^2)(t-t')}dt' \qquad (12.5)$$

XIII. NUMERICAL RESULT

Set $\alpha=\frac{\pi}{2}$, a =0.5 m, b = 2 m, ξ =1.5 m and t = 1 sec and λ_n are the roots of the transcendental equation

$$J_0(\lambda_n a)Y_0(\lambda_n b)-J_0(\lambda_n b)Y_0(\lambda_n a)=0$$

as [3] in (12.4) one obtains

$$\frac{g(z,t)}{\alpha}=\sum_{m=1}^\infty e^{-p}\sin(3.14mz)\ \left\{\sum_{n=1}^\infty \frac{\lambda_n^2 J_0(.1\lambda_n)J_0(.15\lambda_n)}{(\lambda_n^2+9.87m^2)}\right.$$

$$\times\left[\frac{(.15)J_0(.1\lambda_n)Y_0(.2\lambda_n)-J_0(.2\lambda_n)Y_0(.1\lambda_n)}{J_0{}^2(.1\lambda_n)-J_0{}^2(.15\lambda_n)}\right]$$

$$-\sum_{n=1}^{\infty}\frac{\lambda_n{}^2 J^2{}_0(.15\lambda_n)}{(\lambda_n{}^2+9.87m^2)}$$

$$\times\left[\frac{(.1)J_0(.15\lambda_n)Y_0(.2\lambda_n)+J_0(.2\lambda_n)Y_0(.15\lambda_n)}{J_0{}^2(.1\lambda_n)-J_0{}^2(.15\lambda_n)}\right]\Bigg\}$$

$$\times\int_0^1(1-e^{-t'})e^{-0.86(\lambda_n{}^2+9.87m^2)(1-t')}dt' \tag{13.1}$$

XIV. CONCLUSION

In both the problems, the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of annular beam have been derived. The Fourier sine transform and Laplace transform techniques are used to obtain the numerical results. The results that are obtained can be applied to the design of useful structures or machines in engineering applications. Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions.

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