

Thermal Stresses of Semi-Infinite Rectangular Plate: Steady-State Problem

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Abstract- This paper is concerned with steady-state thermoelastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite rectangular plate when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.

Key Words: Semi-infinite rectangular plate, steady-state problem, Integral transform, direct problem

I. INTRODUCTION

In 1999, **Adams and Bert** [1] studied thermoelastic vibrations of a laminated rectangular plate subjected to a thermal shock. **Tanigawa and Komatsubara** [2] discussed thermal stress analysis of a rectangular plate and its thermal stress intensity factor for compressive stress field. **Vihak; Yuzvyak and Yasinskij** [3]: derived the solution of the plane thermoelasticity problem for a rectangular domain. **Dange; Khobragade and Durge** [4] studied three dimensional inverse transient thermoelastic problem of a thin rectangular plate. **Ghume and Khobragade** [5] investigated deflection of a thick rectangular plate. **Roy and Khobragade** [6] discussed transient thermoelastic problem of an infinite rectangular slab. **Lamba and Khobragade** [7] studied thermoelastic problem of a thin rectangular plate due to partially distributed heat supply.

In 2012, **Sutar and Khobragade** [8] discussed inverse thermoelastic problem of heat conduction with internal heat generation for the rectangular plate. **Khobragade; Hiranwar; and Khalsa** [9] derived thermal deflection of a thick clamped rectangular plate. **Roy; Bagade and Khobragade** [10] studied thermal stresses of a semi infinite rectangular beam. **Jadhav and Khobragade** [11] discussed inverse thermoelastic problem of a thin finite rectangular plate due to internal heat source. **Singru and Khobragade** [12] studied thermal stress analysis of a thin rectangular plate with internal heat source. **Further Singru and Khobragade** [13] derived, Thermal stresses of a semi-infinite rectangular slab with internal heat generation.

In this paper, an attempt has been made to solve two direct steady-state problems of thermoelasticity.

In the first problem, an attempt has been made to determine the temperature distribution, displacement function and thermal stresses functions of semi-infinite rectangular plate

occupying the space $D: 0 \leq x \leq a, 0 \leq y \leq \infty$ with known boundary conditions.

In the second problem, an attempt has been made to determine the temperature distribution, displacement function and thermal stresses of semi-infinite rectangular plate occupying the space $D: 0 \leq x \leq a, 0 \leq y \leq \infty$ with known boundary conditions.

II. STATEMENT OF THE PROBLEM-I

Consider semi-infinite rectangular plate occupying the space $D: 0 \leq x \leq a, 0 \leq y \leq \infty$. The displacement components u_x and u_y in the x and y - direction represented in the integral form as [2] are

$$u_x = \int \left[\frac{1}{E} \left(\frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \quad (2.1)$$

$$u_y = \int \left[\frac{1}{E} \left(\frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \quad (2.2)$$

where ν and α are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate respectively and $U(x,y)$ is the Airy's stress function which satisfy the following relation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 U = -\alpha E \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T \quad (2.3)$$

where E is the Young's modulus of elasticity and T is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (2.4)$$

subject to the boundary conditions

$$T(0, y) = 0 \quad (2.5)$$

$$T(a, y) = f(y) \quad (2.6)$$

$$\left[\frac{dT(x, y)}{dy} \right]_{y=0} = 0 \quad (2.7)$$

$$\left[\frac{dT(x, y)}{dy} \right]_{y=\infty} = 0 \quad (2.8)$$

The stress components in terms of U are given by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \quad (2.9)$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \quad (2.10)$$

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \quad (2.11)$$

Equations (2.1) to (2.11) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying Fourier cosine transform to the equations (2.4), (2.5), (2.6) and using the conditions (2.7), (2.8) one obtains

$$\frac{d^2 \bar{T}_c}{dx^2} - p^2 \bar{T}_c = 0 \quad (3.1)$$

$$\text{where } p^2 = m^2 \pi^2 \quad (3.2)$$

$$\bar{T}_c(0, m) = 0 \quad (3.3)$$

$$\bar{T}_c(a, m) = \bar{f}_c(m) \quad (3.4)$$

where \bar{T}_c denotes Fourier cosine transform of T and m is cosine transform parameter.

Equation (3.1) is a second order differential equation whose solution gives

$$\bar{T}_c(x, m) = Ae^{px} + Be^{-px} \quad (3.5)$$

where A, B are arbitrary constants.

Using (3.3) and (3.4) in (3.5) one obtains

$$A + B = 0 \quad (3.6)$$

$$Ae^{pa} + Be^{-pa} = \bar{f}_c(m) \quad (3.7)$$

Solving (3.6) and (3.7) one obtains

$$A = \frac{\bar{f}_c(m)}{e^{pa} - e^{-pa}}, \quad B = -\frac{\bar{f}_c(m)}{e^{pa} - e^{-pa}}$$

Substituting the values of A and B in (3.5) one obtains

$$\bar{T}_c(x, m) = \bar{f}_c(m) \frac{\sinh(px)}{\sinh(pa)} \quad (3.8)$$

Applying inverse Fourier cosine transform to the equations (3.8) one obtains the expression for temperature distribution T(x,y) as

$$T(x, y) = \frac{1}{\pi} \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \quad (3.9)$$

$$\text{where } \bar{f}_c(m) = \int_0^{\infty} f(y) \cos py \, dy$$

Substituting the value of T(x,y) from (3.9) in (2.1) one obtains the expression for Airy's stress function U(x,y) as

$$U(x, y) = -\frac{\alpha E}{\pi p^2} \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \quad (3.10)$$

IV. DETERMINATION OF THERMOELASTIC DISPLACEMENT

Substituting the value of U(x,y) from (3.10) in (2.1) and (2.2) one obtains the thermoelastic displacement functions u_x and u_y as

$$u_x = \left[\frac{\alpha(2+\nu)}{\pi p} \right] \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\cosh(px)}{\sinh(pa)} \right] \quad (4.1)$$

$$u_y = \left[\frac{-\alpha\nu}{\pi p} \right] \sum_{m=1}^{\infty} \bar{f}_c(m) \sinh(py) \left[\frac{\sinh(px)}{\sinh(pa)} \right] \quad (4.2)$$

V. DETERMINATION OF STRESS FUNCTIONS

Using (3.10) in (2.9), (2.10) and (2.11) the stress functions are obtained as

$$\sigma_{xx} = \left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \quad (5.1)$$

$$\sigma_{yy} = -\left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \quad (5.2)$$

$$\sigma_{xy} = \left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{f}_c(m) \sin py \left[\frac{\cosh(px)}{\sinh(pa)} \right] \quad (5.3)$$

VI. SPECIAL CASE

$$\text{Set } f(y) = e^{-y^2} a \tag{6.1}$$

Applying finite Fourier cosine transform to the equation (6.1) one obtains

$$\begin{aligned} \bar{f}_c(m) &= \int_0^{\infty} (1+y)e^{-y^2} a \cos(py) dy \\ &= \left(\frac{a\sqrt{\pi}e^{-p^2/4}}{2} \right) \end{aligned} \tag{6.2}$$

Substituting the value of $\bar{f}_c(m)$ from (6.2) in the equation (3.9) one obtains

$$T(x, y) = \left(\frac{\xi}{2\sqrt{\pi}} \right) \sum_{m=1}^{\infty} e^{-p^2/4} \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \tag{6.3}$$

VII. NUMERICAL RESULT

Set $\beta = \frac{a}{2\sqrt{\pi}}$, $\pi = 3.14$, $a = 1.5$ m in equation (6.3) to obtain

$$\frac{T(x, y)}{\beta} = \sum_{m=1}^{\infty} e^{-p^2/4} \cos(1.57my) \left[\frac{\sinh(3.14m)}{\sinh(1.36m)} \right] \tag{7.1}$$

VIII. STATEMENT OF THE PROBLEM-II

Consider semi-infinite rectangular plate occupying the space $D : 0 \leq x \leq a, 0 \leq y \leq \infty$. The displacement components u_x and u_y in the x and y- direction represented in the integral form as [2] are

$$u_x = \int \left[\frac{1}{E} \left(\frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \tag{8.1}$$

$$u_y = \int \left[\frac{1}{E} \left(\frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \tag{8.2}$$

where ν and α are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate respectively and $U(x,y)$ is the Airy's stress function which satisfy the following relation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 U = -\alpha E \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T \tag{8.3}$$

where E is the Young's modulus of elasticity and T is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{8.4}$$

subject to the boundary conditions

$$T(0, y) = h(y) \tag{8.5}$$

$$T(a, y) = f(y) \tag{8.6}$$

$$\left[\frac{dT(x, y)}{dy} \right]_{y=0} = 0 \tag{8.7}$$

$$\left[\frac{dT(x, y)}{dy} \right]_{y=\infty} = 0 \tag{8.8}$$

The stress components in terms of U are given by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \tag{8.9}$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \tag{8.10}$$

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \tag{8.11}$$

Equations (8.1) to (8.11) constitute the mathematical formulation of the problem under consideration.

IX. SOLUTION OF THE PROBLEM

Applying Fourier cosine transform to the equations (8.4), (8.5) (8.6) and using (8.7), (8.8) one obtains

$$\frac{d^2 \bar{T}_c}{dx^2} - p^2 \bar{T}_c = 0 \tag{9.1}$$

$$\text{where } p^2 = m^2 \pi^2 \tag{9.2}$$

$$\bar{T}_c(0, m) = \bar{h}_c(m) \tag{9.3}$$

$$\bar{T}_c(a, m) = \bar{f}_c(m) \tag{9.4}$$

where \bar{T}_c denotes Fourier cosine transform of T and m is cosine transform parameter.

The equation (9.1) is a second order differential equation whose solution gives

$$\bar{T}_c(x, m) = Ae^{px} + Be^{-px} \quad (9.5)$$

where A, B are arbitrary constants.

Using (9.3) and (9.4) in (9.5) one obtains

$$A + B = \bar{h}_c(m) \quad (9.6)$$

$$Ae^{p\xi} + Be^{-p\xi} = \bar{f}_c(m) \quad (9.7)$$

Solving (9.6) and (9.7) one obtains

$$A = \frac{\bar{f}_c(m)}{e^{p\xi} - e^{-p\xi}} - \frac{\bar{h}_c(m)e^{-p\xi}}{e^{p\xi} - e^{-p\xi}}$$

$$B = -\frac{\bar{f}_c(m)}{e^{p\xi} - e^{-p\xi}} + \frac{\bar{h}_c(m)e^{p\xi}}{e^{p\xi} - e^{-p\xi}}$$

Substituting the values of A and B in (9.5) one obtains

$$\bar{T}_c(x, m) = \bar{f}_c(m) \left[\frac{\sinh(px)}{\sinh(pa)} \right] - \bar{h}_c(m) \left[\frac{\sinh(p(x-a))}{\sinh(pa)} \right] \quad (9.8)$$

Applying inverse Fourier cosine transform to the equations (9.8) one obtain the expression for temperature distribution T(x,y) as

$$T(x, y) = \frac{1}{\pi} \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right]$$

$$- \frac{1}{\pi} \sum_{m=1}^{\infty} \bar{h}_c(m) \cos py \left[\frac{\sinh(p(x-a))}{\sinh(pa)} \right] \quad (9.9)$$

where $\bar{f}_c(m) = \int_0^b f(y) \sin py dy$,

$$\bar{h}_c(m) = \int_0^b h(y) \sin py dy$$

Substituting the value of T(x,y) from (9.9) in (8.3) one obtains the expression for Airy's stress function U(x,y) as

$$U(x, y) = -\frac{\alpha E}{\pi p^2} \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right]$$

$$+ \frac{2\alpha E}{\pi p^2} \sum_{m=1}^{\infty} \bar{h}_s(m) \cos py \left[\frac{\sinh(p(x-a))}{\sinh(pa)} \right] \quad (9.10)$$

X. DETERMINATION OF THERMOELASTIC DISPLACEMENT

Substituting the value of U(x,y) from (9.10) in (8.1) and (8.2) one obtains the thermoelastic displacement functions u_x and u_y as

$$u_x = \left[\frac{2\alpha(2+\nu)}{\pi} \sum_{m=1}^{\infty} \bar{f}_c(m) \left[\frac{\sin py}{\sinh(pa)} \right] \left[\frac{\cosh(pa) - 1}{m} \right] \right.$$

$$\left. - \left[\frac{2\alpha(2+\nu)}{\pi} \sum_{m=1}^{\infty} \bar{h}_c(m) \left[\frac{\sin py}{\sinh(pa)} \right] \left[\frac{1 - \cosh(pa)}{m} \right] \right] \quad (10.1)$$

$$u_y = \left[\frac{2\alpha(2+\nu)}{\pi} \sum_{m=1}^{\infty} \bar{f}_c(m) \left[\frac{\sinh(px)}{\sinh(pa)} \right] \left[\frac{\cos(pa) - 1}{m} \right] \right.$$

$$\left. - \left[\frac{2\alpha(2+\nu)}{\pi} \sum_{m=1}^{\infty} \bar{h}_c(m) \left[\frac{\sinh(p(x-a))}{\sinh(pa)} \right] \left[\frac{\cos(pa) - 1}{m} \right] \right] \quad (10.2)$$

XI. DETERMINATION OF STRESS FUNCTIONS

Using (9.10) in (8.9) , (8.10) and (8.11) the stress functions are obtained as

$$\sigma_{xx} = \left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right]$$

$$- \left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{h}_c(m) \cos py \left[\frac{\sinh(p(x-a))}{\sinh(pa)} \right] \quad (11.1)$$

$$\sigma_{yy} = -\left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{f}_c(m) \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right]$$

$$+ \left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{h}_c(m) \cos py \left[\frac{\sinh(p(x-a))}{\sinh(pa)} \right] \quad (11.2)$$

$$\sigma_{xy} = \left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{f}_s(m) \cos py \left[\frac{\cosh(px)}{\sinh(pa)} \right]$$

$$- \left(\frac{\alpha E}{\pi} \right) \sum_{m=1}^{\infty} \bar{h}_s(m) \cos py \left[\frac{\cosh(p(x-a))}{\sinh(pa)} \right] \quad (11.3)$$

XII. SPECIAL CASE

$$\text{Set } f(y) = e^{-y^2} e^a, \quad h(y) = e^{-y^2} \quad (12.1)$$

Applying Fourier cosine transform to the equation (12.1) one obtains

$$\begin{aligned} \bar{f}_c(m) &= \int_0^{\infty} e^{-y^2} e^a \cos py \, dy \\ &= \left(\frac{\sqrt{\pi} e^{-p^2/4} e^a}{2} \right) \end{aligned} \quad (12.2)$$

$$\begin{aligned} \bar{h}_c(m) &= \int_0^{\infty} e^{-y^2} \cos py \, dy \\ &= \left(\frac{\sqrt{\pi} e^{-p^2/4}}{2} \right) \end{aligned} \quad (12.3)$$

Substituting the values of $\bar{f}_s(m)$ and $\bar{h}_s(m)$ from (12.2) and (12.3) in the equations (9.9) one obtains

$$\begin{aligned} T(x, y) &= \frac{e^a}{2\sqrt{\pi}} \sum_{m=1}^{\infty} e^{-p^2/4} \cos py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \\ &\quad - \frac{1}{2\sqrt{\pi}} \sum_{m=1}^{\infty} e^{-p^2/4} \cos py \left[\frac{\sinh(p(x-a))}{\sinh(pa)} \right] \end{aligned} \quad (12.4)$$

XIII. NUMERICAL RESULT

Set $\beta = \frac{1}{2\sqrt{\pi}}$, $\pi = 3.14$, $a = 1.5$ m in the equation (12.4) to obtain

$$\frac{g(y)}{\beta} = \sum_{m=1}^{\infty} \cos(1.57my) \left\{ \begin{array}{l} \left[\frac{\sinh(3.14m)}{\sinh(1.36m)} \right] (e^{1.5}) \\ - \left[\frac{\sinh(0.79m)}{\sinh(1.36m)} \right] \end{array} \right\} \quad (13.1)$$

XIV. CONCLUSION

In both the problems, the temperature distribution, displacement function and thermal stresses of semi-infinite rectangular beam have been investigated with the aid of integral transform techniques. The expressions are obtained in terms of Bessel's function in the form of infinite series. The results that are obtained can be applied to the design of useful structures or machines in engineering applications.

Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions.

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