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Solution of the Homogeneous and Non homogeneous Diffusion heat equation by Variational Homotopy Perturbation Method

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Abstract: **The Variational Homotopy Perturbation Method (VHPM) deforms a difficult problem into a simple problem which can be easily solved. In this work VHPM is applied to solve homogeneous and non homogeneous Diffusion equations. The obtained results are found to be in good agreement with the exact solutions known**.

Key words**: VHPM, Diffusion equations, Boundary conditions.**

I. INTRODUCTION

In the recent years, the application of Homotopy Perturbation Method (HPM) in nonlinear problems has been developed by many Mathematicians and engineers to solve various differential equations problems. The HPM deforms the difficult problem under study into a simple problem which is easy to solve. The HPM and the VIM (Variational Iteration Method) was introduced by Ji-Huan He [1-6] of Shanghai University and was further improved by Ganji[7-8], Yang[9-10], Zhang[11] and so on. The VHPM is based on the HPM and the VIM. The method employs a homotopy transform to generate a convergent series solution of differential equations.

II. BASIC IDEA OF VARIATIONAL HOMOTOPY PERTURBATION METHOD

Consider the nonlinear Differential equation:

$$L\left[u(x,t)\right]+N\left[u(x,t)\right]=g(t) \tag{1}$$

Where L = Linear operator

N = Nonlinear operator

$g(t)$ = Analytical function(Known)

According to Variational Iteration method(VIM), we write a correctional functional as

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^x \lambda(x,s)\left[Lu_n(x,s)+N\tilde{u}_n(x,s)-g(s)\right]ds \tag{2}$$

Where λ = general Lagrangian multiplier which can be identified optimally,

u_n = nth approximate solution,

$\tilde{u}_n$ = restricted variation, i.e., $\delta\tilde{u}_n$ =0.

By Homotopy perturbation method(HPM), we can construct an equation is as follows

$$\sum_{i=0}^{\infty} p^i u_i = u_0(x)+p\int_0^x \lambda(x,s)\left[\sum_{i=0}^{\infty} L\left(p^i u_i(x,s)\right)+N\left(p^i\tilde{u}_i(x,s)\right)\right]ds-\int_0^x \lambda(x,s)g(s)ds \tag{3}$$

Usually an approximation to the solution will be obtained by identical powers of p and taking the limit as $p\to 1$, we get

$$u(x,t)=\lim_{p\to 1}\sum_{i=0}^{\infty} p^i u_i(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\dots\dots\dots \tag{4}$$

III. HOMOGENEOUS DIFFUSION EQUATION

Consider Linear homogeneous Diffusion Equation

$$u_t=u_{xx}-u,\ 0<x<1,\ t>0 \tag{5}$$

Boundary conditions are given by

$$u(0,t)=0=u(1,t),\ t>0 \tag{6}$$

Initial condition is given by

$$u(x,0)=\sin\pi x,\ 0<x<1 \tag{7}$$

This is a Heat Equation which is solved by VHPM.

By VHPM, consider

$$L(u)=u_t \text{ and } N(u)=u_{xx}-u \tag{8}$$

Where L is a linear operator and N is a nonlinear operator. In order to construct a correction functional for this system, we can write the following expression

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda(x,s)\left[\left(u_n(x,s)\right)_t-\left(\tilde{u}_n(x,s)\right)_{xx}+\left(\tilde{u}_n(x,s)\right)\right]ds \tag{9}$$

Where,

$\tilde{u}_n$ = restricted variation, i.e., $\delta\tilde{u}_n$ =0.

To find the optimal value of λ ,we make the correction functional(9) stationary in the following form

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t\lambda(x,s)\left[\left(u_n(x,s)\right)_t-\left(\tilde{u}_n(x,s)\right)_{xx}+\left(\tilde{u}_n(x,s)\right)\right]ds$$

$$=\delta u_n(x,t)+\lambda(x,s)\delta u_n(x,s)\Big|_{s=t}$$

$$-\int_0^t\frac{\partial}{\partial s}\lambda(x,s)\delta u_n(x,s)ds-\int_0^t\frac{\partial}{\partial s}\lambda(x,s)-\delta\left(\tilde{u}_n(x,s)\right)_{xx}+\delta\left(\tilde{u}_n(x,s)\right)ds$$

$$=1+\lambda(x,s)\delta u_n(x,s)\Big|_{s=t}-\int_0^t\frac{\partial}{\partial s}\lambda(x,s)\delta u_n(x,s)ds=0$$

Hence, we have the following stationary conditions

$$\frac{\partial}{\partial s}\lambda(x,s)\Big|_{s=t}=0$$

$$1+\lambda(x,s)\Big|_{s=t}=0$$

$$\Rightarrow\lambda(x,s)=-1$$

Substituting this value in equation (2), we get

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left[\left(u_n(x,s)\right)_t-\left(u_n(x,s)\right)_{xx}+\left(u_n(x,s)\right)\right]ds \tag{10}$$

Then equation (9) will enable to determine the components $u_n(x,t)$ recursively for $n\geq 0$.

Now, by exerting the VHPM, it is possible to obtain the equation as follows,

$$u_0+pu_1+p^2u_2+.....=\sin\pi x+p\int_0^t\left[\left(u_0+pu_1+p^2u_2+.....\right)_{xx}\right]ds-p\int_0^t\left[\left(u_0+pu_1+p^2u_2+.....\right)\right]ds$$

Comparing powers of p from both sides, we get

$$p^0:u_0(x,t)=\sin\pi x \tag{11}$$

$$p^1:u_1(x,t)=\int_0^t(u_0)_{xx}ds-\int_0^t(u_0)ds=\int_0^t-\pi^2\sin\pi x ds-\int_0^t\sin\pi x ds=-\left(\pi^2+1\right)t\sin\pi x \tag{12}$$

$$p^2:u_2(x,t)=\int_0^t(u_1)_{xx}ds-\int_0^t(u_1)ds=\int_0^t\left[-\left(\pi^2+1\right)t\sin\pi x\right]_{xx}ds-\int_0^t\left[-\left(\pi^2+1\right)t\sin\pi x\right]ds$$

$$=\left(\pi^2+1\right)^2t^2\sin\pi x \tag{13}$$

$$p^3:u_3(x,t)=\int_0^t(u_2)_{xx}ds-\int_0^t(u_2)ds=\int_0^t\left[-\left(\pi^2+1\right)^2t^2\sin\pi x\right]_{xx}ds-\int_0^t\left[-\left(\pi^2+1\right)^2t^2\sin\pi x\right]ds$$

$$=-\left(\pi^2+1\right)^3t^3\sin\pi x \tag{14}$$

and so on, we get an equation (4) becomes

$$u(x,t)=\sin\pi x-\left(\pi^2+1\right)t\sin\pi x+\left(\pi^2+1\right)^2t^2\sin\pi x-\left(\pi^2+1\right)^3t^3\sin\pi x+.......... \tag{15}$$

$$=\sin\pi x\left[1-\left(\pi^2+1\right)t+\left(\pi^2+1\right)^2t^2-\left(\pi^2+1\right)^3t^3+.........\right]$$

$$=\sin\pi x e^{-\left(\pi^2+1\right)t} \tag{16}$$

This is the exact solution of (5).

IV. NON HOMOGENEOUS DIFFUSION EQUATION

Consider non homogeneous Diffusion Equation

$$u_t-3u_{xx}=x, 0<x<\pi, t>0 \tag{17}$$

Boundary conditions are given by

$$u(0,t)=0, u(\pi,t)=-\pi t \tag{18}$$

Initial condition is given by

$$u(x,0)=\sin x, 0<x<\pi \tag{19}$$

This is a Heat equation which is solved by VHPM

By VHPM, consider

$$L(u)=u_t \text{ and } N(u)=-3u_{xx}-x \tag{20}$$

Now, by exerting the VHPM, it is possible to obtain an equation as follows

$$u_0+pu_1+p^2u_2+p^3u_3+........=\sin x-3p\int_0^t\left[\left(u_0+pu_1+p^2u_2+p^3u_3+........\right)_{xx}\right]ds-p\int_0^t xds \tag{21}$$

$$u_0+pu_1+p^2u_2+p^3u_3+........=\sin x-3p\int_0^t\left[\left(u_0\right)_{xx}+p\left(u_1\right)_{xx}+p^2\left(u_2\right)_{xx}+........\right]ds-p\int_0^t xds \tag{22}$$

Comparing the powers of p from both sides we get the following results

$$p^0:u_0\left(x,t\right)=\sin x \tag{23}$$

$$p^1:u_1\left(x,t\right)=-3\int_0^t\left(\sin x\right)_{xx}ds-\int_0^t xds=3t\sin x-xt \tag{24}$$

$$p^2:u_2\left(x,t\right)=-3\int_0^t\left(u_1\right)_{xx}ds=-3\int_0^t\left(3t\sin x-xt\right)_{xx}ds=9t^2\sin x \tag{25}$$

$$p^3:u_3\left(x,t\right)=-3\int_0^t\left(u_2\right)_{xx}ds=-3\int_0^t\left(9t^2\sin x\right)_{xx}ds=27t^3\sin x \tag{26}$$

and so on, we get an equation (4) becomes

$$u\left(x,t\right)=\sin x+3t\sin x-xt+9t^2\sin x+27t^3\sin x+................$$

$$=\sin x\left[1+3t+9t^2+27t^3+...............\right]-xt$$

$$=\frac{\sin x}{1-3t}-xt$$

This is the exact solution of (17).

V. CONCLUSION

In this paper, VHPM has been successfully applied for homogeneous and non homogeneous diffusion equations and compared with exact solutions. The present method is easy and reliable to use. Thus, VHPM is one of the successful method to solve homogeneous, non homogeneous problems and gives quickly convergent.

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