Thermoelastic Solution of Semi-Infinite Rectangular Plate: Direct Problem

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***Abstract*- This paper is concerned with steady-state thermoelastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of semi-infinite rectangular plate when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.**

***Key Words:* Semi-infinite rectangular plate, direct problem, Integral transform technique**

I. INTRODUCTION

In 1999, **Adams and Bert [1]** studied thermoelastic vibrations of a laminated rectangular plate subjected to a thermal shock. **Tanigawa and Komatsubara [2]** discussed thermal stress analysis of a rectangular plate and its thermal stress intensity factor for compressive stress field. **Vihak; Yuzvyak and Yasinskij [3]**: derived the solution of the plane thermoelasticity problem for a rectangular domain. **Dange; Khobragade and Durge [4]** studied three dimensional inverse transient thermoelastic problem of a thin rectangular plate. **Ghume and Khobragade [5]** investigated deflection of a thick rectangular plate. **Roy and Khobragade [6]** discussed transient thermoelastic problem of an infinite rectangular slab. **Lamba and Khobragade [7]** studied thermoelastic problem of a thin rectangular plate due to partially distributed heat supply.

In 2012**, Sutar and Khobragade [8]** discussed inverse thermoelastic problem of heat conduction with internal heat generation for the rectangular plate. **Khobragade; Hiranwar;** and Khalsa [9] derived thermal deflection of a thick clamped rectangular plate. **Roy; Bagade and Khobragade [10]** studied thermal stresses of a semi infinite rectangular beam. **Jadhav and Khobragade [11]** discussed inverse thermoelastic problem of a thin finite rectangular plate due to internal heat source. **Singru and Khobragade [12]** studied thermal stress analysis of a thin rectangular plate with internal heat source. **Further Singru and Khobragade [13] derived**, Thermal stresses of a semi-infinite rectangular slab with internal heat generation. **Barai; Warbhe and Khobragade [14]** studied inverse steady-state thermoelastic problems of semi-infinite rectangular plate and **Barai; Warbhe and Khobragade [15]** discussed inverse transient thermoelastic problem of semi-infinite rectangular plate.

In this paper, an attempt has been made to discuss two steady-state problems of thermoelasticity. In both the problems, an attempt has been made to determine the temperature distribution, displacement function and thermal stresses functions of semi-infinite rectangular plate occupying the space D: $0 \le x \le a$, $0 \le y \le \infty$ with known boundary conditions.

II. STATEMENT OF THE PROBLEM-I

Consider semi-infinite rectangular plate occupying the space $D: 0 \le x \le a,\ 0 \le y \le \infty$. The displacement components u_x and u_y in the x and y- direction represented in the integral form as [2] are

$$u_x = \int \left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \qquad (2.1)$$

$$u_y = \int \left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \qquad (2.2)$$

where ν and α are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate respectively and U(x,y) is the Airy's stress function which satisfy the following relation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 U = -\alpha E \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T \qquad (2.3)$$

where E is the Young's modulus of elasticity and T is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (2.4)$$

subject to the boundary conditions

$$T(0, y) = 0 \qquad (2.5)$$

$$T(a, y) = f(y)$$

(2.6)

$$T(x,0) = 0 \qquad (2.7)$$

$$T(x,\infty) = 0 \qquad (2.8)$$

The stress components in terms of U are given by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \qquad (2.9)$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \qquad (2.10)$$

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \qquad (2.11)$$

Equations (2.1) to (2.11) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying Fourier sine transform to the equations (2.4), (2.5), (2.6) and using the conditions (2.7), (2.8) one obtains

$$\frac{d^2 \overline{T}_s}{dx^2} - p^2 \overline{T}_s = 0 \qquad (3.1)$$

where $p^2 = m^2 \pi^2$ (3.2)

$$\overline{T}_s(0, m) = 0 \qquad (3.3)$$

$$\overline{T}_s(a, m) = \overline{f}_s(m) \qquad (3.4)$$

where $\overline{T}_S$ denotes Fourier sine transform of T and m is sine transform parameter.

Equation (3.1) is a second order differential equation whose solution gives

$$\overline{T}_s \ (x, m) = Ae^{px} + Be^{-px} \qquad (3.5)$$

where A, B are arbitrary constants.

Using (3.3) and (3.4) in (3.5) one obtains

A + B = 0 (3.6)

$$Ae^{pa} + Be^{-pa} = \overline{f}_s(m) \qquad (3.7)$$

Solving (2.2.7) and (2.2.8) one obtains

$$A = \frac{\overline{f}_s(m)}{e^{pa} - e^{-pa}} \ , \ B = -\frac{\overline{f}_s(m)}{e^{pa} - e^{-pa}}$$

Substituting the values of A and B in (3.5) one obtains

$$\overline{T}_s(x, m) = \overline{f}_s(m) \frac{\sinh(px)}{\sinh(pa)} \qquad (3.8)$$

Applying inverse Fourier sine transform to the equations (3.8) one obtains the expression for temperature distribution T(x,y) as

$$T(x, y) = \frac{1}{\pi} \sum_{m=1}^{\infty} \overline{f}_s(m) \sin py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \qquad (3.9)$$

where $\overline{f}_s(m) = \int_0^\infty f(y) \sin py \, dy$

Substituting the value of T(x,y) from (3.9) in (3.1) one obtains the expression for Airy's stress function U(x,y) as

$$U(x, y) = -\frac{\alpha E}{\pi p^2} \sum_{m=1}^{\infty} \overline{f}_s(m) \sin py \left[\frac{\sinh(px)}{\sinh(pa)} \right] \qquad (3.10)$$

IV. THERMOELASTIC DISPLACEMENT FUNCTIONS

Substituting the value of U(x,y) from (3.10) in (2.1) and (2.2) one obtains the expression for thermoelastic displacement functions u_x and u_y as

$$u_x = \left[\frac{2\alpha(2+\nu)}{\pi} \right] \sum_{m=1}^{\infty} \overline{f}_s(m) \left[\frac{\sin py}{\sinh(pa)} \right] \times \left[\frac{\cosh(pa) - 1}{m} \right] \qquad (4.1)$$

$$u_y = \left[\frac{2\alpha(2+\nu)}{\pi} \right] \sum_{m=1}^{\infty} \overline{f}_s(m) \left[\frac{\sinh(px)}{\sinh(pa)} \right] \times \left[\frac{\cos(pa) - 1}{m} \right] \qquad (4.2)$$

V. STRESS FUNCTIONS

Using (3.10) in (2.9), (2.10) and (2.11), the stress functions are obtained as

$$\sigma_{xx} = \left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty}\overline{f}_s(m)\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right] \quad (5.1)$$

$$\sigma_{yy} = -\left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty}\overline{f}_s(m)\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right] \quad (5.2)$$

$$\sigma_{xy} = \left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty}\overline{f}_s(m)\cos py\left[\frac{\cosh(px)}{\sinh(pa)}\right] \quad (5.3)$$

VI. SPECIAL CASE

Set $f(y) = \left(\frac{y}{1+y^2}\right)a$ (6.1)

Applying Fourier sine transform to the equation (6.1) one obtains

$$\overline{f}_s(m) = \int_0^{\infty}\left(\frac{y}{1+y^2}\right)a\sin(py)dy$$

$$= \left(\frac{\pi a}{2}\right)\left[e^{-p}\right] \quad (6.2)$$

Substituting the value of $\overline{f}_s(m)$ from (6.2) in the equations (3.9), one obtains

$$T(x,y) = \left(\frac{\pi a}{2}\right)\sum_{m=1}^{\infty}\left[e^{-p}\right]\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right] \quad (6.3)$$

VII. NUMERICAL RESULTS

Set $\beta = \left(\frac{\pi a}{2}\right)$, π = 3.14, a = 2 m, in equation (6.3) to obtain

$$\frac{T(x,y)}{\beta} = \sum_{m=1}^{\infty}e^{-p}\sin py\left[\frac{\sinh(px)}{\sinh(2p)}\right] \quad (7.1)$$

VIII. STATEMENT OF THE PROBLEM-II

Consider semi-infinite rectangular plate occupying the space D : $0 \le x \le a$, $0 \le y \le \infty$. The displacement components u_x and u_y in the x and y- direction represented in the integral form as [2] are

$$u_x = \int\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2} - \nu\frac{\partial^2 U}{\partial x^2}\right) + \alpha T\right]dx \quad (8.1)$$

$$u_y = \int\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2} - \nu\frac{\partial^2 U}{\partial y^2}\right) + \alpha T\right]dy \quad (8.2)$$

where ν and α are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate respectively and U(x,y) is the Airy's stress function which satisfy the following relation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 U = -\alpha E\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T \quad (8.3)$$

where E is the Young's modulus of elasticity and T is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (8.4)$$

subject to the boundary conditions

$$T(0,y) = h(y) \quad (8.5)$$

$$T(a,y) = f(y) \quad (8.6)$$

$$T(x,0) = 0 \quad (8.7)$$

$$T(x,\infty) = 0 \quad (8.8)$$

The stress components in terms of U are given by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \quad (8.9)$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \quad (8.10)$$

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \quad (8.11)$$

Equations (8.1) to (8.11) constitute the mathematical formulation of the problem under consideration.

IX. SOLUTION OF THE PROBLEM

Applying Fourier sine transform to the equations (8.4), (8.5) and (8.6) and using (8.7), (8.8) one obtains

$$\frac{d^2\overline{T}_s}{dx^2} - p^2\overline{T}_s = 0 \quad (9.1)$$

where $p^2 = m^2\pi^2$ (9.2)

$$\bar{T}_s(0,m) = \bar{h}_s(m) \tag{9.3}$$

$$\bar{T}_s(a,m) = \bar{f}_s(m) \tag{9.4}$$

where $\bar{T}_S$ denotes Fourier sine transform of T and m is sine transform parameter.

Equation (9.1) is a second order differential equation whose solution gives

$$\bar{T}_s\ (x,m) = Ae^{px} + Be^{-px} \tag{9.5}$$

where A, B are arbitrary constants.

Using (9.3) and (9.4) in (9.5) one obtains

$$A + B = \bar{h}_s(m) \tag{9.6}$$

$$Ae^{pa} + Be^{-pa} = \bar{f}_s(m) \tag{9.7}$$

Solving (9.6) and (9.7) we get

$$A = \frac{\bar{f}_s(m)}{e^{pa} - e^{-pa}} - \frac{\bar{h}_s(m)e^{-pa}}{e^{pa} - e^{-pa}}$$

$$B = -\frac{\bar{f}_s(m)}{e^{pa} - e^{-pa}} + \frac{\bar{h}_s(m)e^{pa}}{e^{pa} - e^{-pa}}$$

Substituting the values of A and B in (9.5) one obtains

$$\bar{T}_s(x,m) = \bar{f}_s(m)\frac{\sinh(px)}{\sinh(pa)} - \bar{h}_s(m)\frac{\sinh(p(x-a))}{\sinh(pa)} \tag{9.8}$$

Applying inverse Fourier sine transform to the equations (9.8) one obtains the expression for temperature distribution T(x,y) as

$$T(x,y) = \frac{1}{\pi}\sum_{m=1}^{\infty}\bar{f}_s(m)\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right] - \frac{1}{\pi}\sum_{m=1}^{\infty}\bar{h}_s(m)\sin py\left[\frac{\sinh(p(x-a))}{\sinh(pa)}\right] \tag{9.9}$$

where $\bar{f}_s(m) = \int_0^{\infty} f(y)\sin py\,dy$,

$$\bar{h}_s(m) = \int_0^{\infty} h(y)\sin py\,dy$$

Substituting the value of T(x,y) from (9.9) in (8.3) one obtains the expression for Airy's stress function U(x,y) as

$$U(x,y) = -\frac{\alpha E}{\pi p^2}\sum_{m=1}^{\infty}\bar{f}_s(m)\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right] + \frac{\alpha E}{\pi p^2}\sum_{m=1}^{\infty}\bar{h}_s(m)\sin py\left[\frac{\sinh(p(x-a))}{\sinh(pa)}\right] \tag{9.10}$$

X. THERMOELASTIC DISPLACEMENT FUNCTIONS

Substituting the value of U(x,y) from (9.10) in (8.1) and (8.2) one obtains the thermoelastic displacement functions u_x and u_y as

$$u_x = \left[\frac{\alpha(2+\nu)}{\pi}\right]\sum_{m=1}^{\infty}\bar{f}_s(m)\left[\frac{\sin py}{\sinh(pa)}\right]\left[\frac{\cosh(pa)-1}{m}\right] - \left[\frac{\alpha(2+\nu)}{\pi}\right]\sum_{m=1}^{\infty}\bar{h}_s(m)\left[\frac{\sin py}{\sinh(pa)}\right]\left[\frac{1-\cosh(pa)}{m}\right] \tag{10.1}$$

$$u_y = \left[\frac{\alpha(2+\nu)}{\pi}\right]\sum_{m=1}^{\infty}\bar{f}_s(m)\left[\frac{\sinh(px)}{\sinh(pa)}\right]\left[\frac{\cos(pa)-1}{m}\right] - \left[\frac{\alpha(2+\nu)}{\pi}\right]\sum_{m=1}^{\infty}\bar{h}_s(m)\left[\frac{\sinh(p(x-a))}{\sinh(pa)}\right]\left[\frac{\cos(pa)-1}{m}\right] \tag{10.2}$$

XI. STRESS FUNCTIONS

Using (9.10) in (8.9), (8.10) and (8.11) the stress functions are obtained as

$$\sigma_{xx} = \left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty}\bar{f}_s(m)\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right] - \left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty}\bar{h}_s(m)\sin py\left[\frac{\sinh(p(x-a))}{\sinh(pa)}\right] \tag{11.1}$$

$$\sigma_{yy} = -\left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty}\bar{f}_s(m)\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right] + \left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty}\bar{h}_s(m)\sin py\left[\frac{\sinh(p(x-a))}{\sinh(pa)}\right] \tag{11.2}$$

$$\sigma_{xy} = \left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty} \bar{f}_s(m)\cos py\left[\frac{\cosh(px)}{\sinh(pa)}\right]$$
$$-\left(\frac{\alpha E}{\pi}\right)\sum_{m=1}^{\infty} \bar{h}_s(m)\cos py\left[\frac{\cosh(p(x-a))}{\sinh(pa)}\right] \quad (11.3)$$

XII. SPECIAL CASE

Set $f(y) = \left(\frac{y}{1+y^2}\right)e^a, h(y) = \left(\frac{y}{1+y^2}\right)$ (12.1)

Applying Fourier sine transform to the equation (12.1) one obtains

$$\bar{f}_s(m) = \int_0^{\infty}\left(\frac{y}{1+y^2}\right)e^a \sin(py)dy$$
$$= \left(\frac{\pi e^a}{2}\right)\left[e^{-p}\right] \quad (12.2)$$

$$\bar{h}_s(m) = \int_0^{\infty}\left(\frac{y}{1+y^2}\right)\sin(py)dy$$
$$= \left(\frac{\pi}{2}\right)\left[e^{-p}\right] \quad (12.3)$$

Substituting the values of $\bar{f}_s(m)$ and $\bar{h}_s(m)$ from (12.2) and (12.3) in the equations (9.9) one obtains

$$T(x,y) = \left(\frac{e^a}{2}\right)\sum_{m=1}^{\infty}\left[e^{-p}\right]\sin py\left[\frac{\sinh(px)}{\sinh(pa)}\right]$$
$$-\left(\frac{1}{2}\right)\sum_{m=1}^{\infty}\left[e^{-p}\right]\sin py\left[\frac{\sinh(p(x-a))}{\sinh(pa)}\right] \quad (12.4)$$

XIII. NUMERICAL RESULTS

Set $\beta = \frac{1}{2}$, π = 3.14, a = 2 m, in the equation (12.4) to obtain

$$\frac{T(x,y)}{\beta} = \sum_{m=1}^{\infty}\left[e^{-p}\right]\sin py\left[\frac{\sinh(px)}{\sinh(2p)}\right]$$
$$-\sum_{m=1}^{\infty}\left[e^{-p}\right]\sin py\left[\frac{\sinh(p(x-2))}{\sinh(2p)}\right] \quad (12.4)$$

$$\frac{T(x,y)}{\beta} = \sum_{m=1}^{\infty}\left[e^{-p}\right]\sin(py)$$
$$\times\left\{\left[\frac{\sinh(px)}{\sinh(2p)}\right](e^2) - \left[\frac{\sinh(p(x-2))}{\sinh(2p)}\right]\right\}$$
(13.1)

XIV. CONCLUSION

In both the problems, the temperature distribution, displacement function and thermal stresses of semi-infinite rectangular plate have been investigated with the aid of integral transform techniques. The expressions are obtained in terms of Bessel's function in the form of infinite series. The results that are obtained can be applied to the design of useful structures or machines in engineering applications.

Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions

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