Solution of Linear Volterra Integro-Differential Equations of Second Kind Using Mahgoub Transform

Sudhanshu Aggarwal[1], Nidhi Sharma[2], Raman Chauhan[3]
[1]*Assistant Professor, Department of Mathematics, National P.G. College Barhalganj, Gorakhpur-273402, U.P., India*
[2, 3]*Assistant Professor, Department of Mathematics, Noida Institute of Engineering & Technology, Greater Noida-201306, U.P., India*

Abstract: **In this paper, Mahgoub transform is used for solving linear Volterra integro-differential equations of second kind. The technique is described and illustrated with some numerical applications. The results assert that this scheme give the exact results using very less computational work.**

Keywords: **Linear Volterra integro-differential equation, Mahgoub transform, Convolution theorem, Inverse Mahgoub transform.**

I. INTRODUCTION

Mathematical modeling of real life problems usually results in functional equations e.g. stochastic equations, integral and integro-differential equations, partial differential equations and others. In particular integro-differential equations arise in many scientific and engineering applications such as glass forming process, diffusion process, heat transfer, in general neutron diffusion, nano-hydrodynamics and biological species coexisting together with increasing and decreasing rates of generating and wind ripple in the desert.

Volterra studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterra integro-differential equations [1-5] given in the form

$$u^n(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \dots\dots\dots\dots (1)$$

where $u^n(x) = \frac{d^n u}{dx^n}$. Because the resulted equation (1) combines the differential and integral operators, then it is necessary to define initial conditions $u(0), u'(0), \dots\dots.., u^{(n-1)}(0)$ for the determination of the particular solution $u(x)$ of the Volterra integro-differential equation (1).

Any Volterra integro-differential equation is characterized by the existence of one or more of the derivatives $u'(x), u''(x), u'''(x), \dots\dots.$ outside the integral sign. Volterra integro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule.

The Mahgoub transform of the function $F(t)$is defined as [6]:

$$M\{F(t)\} = v\int_0^\infty F(t)e^{-vt}dt = H(v), t \geq 0, k_1 \leq v \leq k_2$$

where M is Mahgoub transform operator.

The Mahgoub transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Mahgoub transform of the function $F(t)$. Mahgoub and Alshikh [7] used Mahgoub transform for solving partial differential equations. Fadhil [8] discussed the convolution for Kamal and Mahgoub transforms. Taha et. al. [9] gave the dualities between Kamal & Mahgoub integral transforms and some famous integral transforms.

The aim of this work is to establish exact solutions for linear Volterra integro-differential equations of second kind using Mahgoub transform without large computational work.

II. MAHGOUB TRANSFORM OF SOME ELEMENTARY FUNCTIONS [6, 8]:

S.N.	$F(t)$	$M\{F(t)\} = H(v)$
1.	1	1
2.	t	$\frac{1}{v}$
3.	t^2	$\frac{2!}{v^2}$
4.	$t^n, n \geq 0$	$\frac{n!}{v^n}$
5.	e^{at}	$\frac{v}{v-a}$
6.	$sinat$	$\frac{av}{v^2+a^2}$

7.	$cosat$	$\frac{v^2}{v^2+a^2}$
8.	$sinhat$	$\frac{av}{v^2-a^2}$
9.	$coshat$	$\frac{v^2}{v^2-a^2}$

III. MAHGOUB TRANSFORM OF THE DERIVATIVES OF THE FUNCTION $F(t)$ [6, 8, 9]:

If $M\{F(t)\} = H(v)$ then

a) $M\{F'(t)\} = vH(v) - vF(0)$
b) $M\{F''(t)\} = v^2H(v) - vF'(0) - v^2F(0)$
c) $M\{F^{(n)}(t)\} =$
$v^n H(v) - v^n F(0) - v^{n-1}F'(0) \ldots\ldots - vF^{(n-1)}(0)$

IV. CONVOLUTION OF TWO FUNCTIONS [8]

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_0^t F(x)G(t-x)dx$$
$$= \int_0^t F(t-x)G(x)dx$$

V. CONVOLUTION THEOREM FOR MAHGOUB TRANSFORMS [8]

If $M\{F(t)\} = H(v)$ and $K\{G(t)\} = I(v)$ then

$$M\{F(t) * G(t)\} = \frac{1}{v}M\{F(t)\}M\{G(t)\} = \frac{1}{v}H(v)I(v)$$

VI. INVERSE MAHGOUB TRANSFORM

If $M\{F(t)\} = H(v)$ then $F(t)$ is called the inverse Mahgoub transform of $H(v)$ and mathematically it is defined as $F(t) = M^{-1}\{H(v)\}$

where M^{-1} is the inverse Mahgoub transform operator.

VII. INVERSE MAHGOUB TRANSFORM OF SOME ELEMENTARY FUNCTIONS

S. N.	$H(v)$	$F(t) = M^{-1}\{H(v)\}$
1.	1	1
2.	$\frac{1}{v}$	t
3.	$\frac{1}{v^2}$	$\frac{t^2}{2!}$
4.	$\frac{1}{v^n}, n \geq 0$	$\frac{t^n}{n!}$
5.	$\frac{v}{v-a}$	e^{at}
6.	$\frac{v}{v^2+a^2}$	$\frac{sinat}{a}$
7.	$\frac{v^2}{v^2+a^2}$	$cosat$
8.	$\frac{v}{v^2-a^2}$	$\frac{sinhat}{a}$
9.	$\frac{v^2}{v^2-a^2}$	$coshat$

VIII. MAHGOUB TRANSFORM FOR LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND KIND

In this section, we present Mahgoub transform for solving linear Volterra integro-differential equations of second kind given by (1). In this work, we will assume that the kernel $k(x,t)$ of (1) is a difference kernel that can be expressed by difference $(x-t)$. The linear Volterra integro-differential equation of second kind (1) can thus be expressed as

$$\left.\begin{matrix} u^n(x) = f(x) + \lambda \int_0^x (x-t)u(t)dt \\ with\, u(0) = a_0, u'(0) = a_1, \ldots, u^{(n-1)}(0) = a_{n-1} \end{matrix}\right\} \ldots.(3)$$

Applying the Mahgoub transform to both sides of(3), we have

$$v^n M\{u(x)\} = v^n a_0 + v^{n-1}a_1 + \cdots \ldots.. + va_{n-1} + M\{f(x)\}$$
$$+ \lambda M\{\int_0^x k(x-t)u(t)dt\} \ldots..(4)$$

Using convolution theorem of Mahgoub transform, we have

$$M\{u(x)\} = a_0 + \frac{a_1}{v} + \cdots \ldots.. + \frac{a_{n-1}}{v^{n-1}} + \frac{1}{v^n}M\{f(x)\}$$
$$+ \frac{\lambda}{v^{n+1}}M\{k(x)\}M\{u(x)\} \ldots\ldots(5)$$

Operating inverse Mahgoub transform on both sides of(5), we have

$$u(x) = a_0 + a_1x + \cdots \ldots. + a_{n-1}\frac{x^{n-1}}{n-1!}$$
$$+ M^{-1}\left\{\frac{1}{v^n}M\{f(x)\}\right\}$$
$$+ \lambda M^{-1}\left\{\frac{1}{v^{n+1}}M\{k(x)\}M\{u(x)\}\right\} \ldots\ldots(6)$$

which is the required solution of (3).

IX. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Mahgoub transform for solving linear Volterra integro-differential equation of second kind.

Application: 1 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u'(x) = 2 + \int_0^x u(t)dt \\ with\, u(0) = 2 \end{array}\right\} \ldots\ldots (7)$$

Applying the Mahgoub transform to both sides of (7) and using initial condition, we have

$$vM\{u(x)\} = 2 + 2v + M\left\{\int_0^x u(t)\,dt\right\} \ldots . (8)$$

Using convolution theorem of Mahgoub transform on (8) and simplify, we have

$$M\{u(x)\} = \frac{2v}{v-1} \ldots \ldots (9)$$

Operating inverse Mahgoub transform on both sides of(9), we have

$$u(x) = M^{-1}\left\{\frac{2v}{v-1}\right\} = 2M^{-1}\left\{\frac{v}{v-1}\right\} = 2e^x \ldots\ldots (10)$$

which is the required exact solution of (7).

Application: 2 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u''(x) = 1 + \int_0^x (x-t)u(t)dt \\ with\, u(0) = 1, u'(0) = 0 \end{array}\right\} \ldots\ldots (11)$$

Applying the Mahgoub transform to both sides of (11) and using initial condition, we have

$$v^2 M\{u(x)\} = 1 + v^2 + M\left\{\int_0^x (x-t)\,u(t)dt\right\} \ldots . (12)$$

Using convolution theorem of Mahgoub transform on (12) and simplify, we have

$$M\{u(x)\} = \frac{v^2}{v^2-1} \ldots \ldots (13)$$

Operating inverse Mahgoub transform on both sides of(13), we have

$$u(x) = M^{-1}\left\{\frac{v^2}{v^2-1}\right\} = coshx \ldots\ldots (14)$$

which is the required exact solution of (11).

Application: 3 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u'''(x) = -1 + \int_0^x u(t)dt \\ with\, u(0) = u'(0) = 1, u''(0) = -1 \end{array}\right\} \ldots\ldots (15)$$

Applying the Mahgoub transform to both sides of (15) and using initial condition, we have

$$v^3 M\{u(x)\} = v^3 + v^2 - v - 1 + \frac{1}{v} M\left\{\int_0^x u(t)\,dt\right\} \ldots . (16)$$

Using convolution theorem of Mahgoub transform on (16) and simplify, we have

$$M\{u(x)\} = \frac{v^2}{v^2+1} + \frac{v}{v^2+1} \ldots \ldots (17)$$

Operating inverse Mahgoub transform on both sides of(17), we have

$$u(x) = M^{-1}\left\{\frac{v^2}{v^2+1}\right\} + M^{-1}\left\{\frac{v}{v^2+1}\right\} = cosx + sinx \ldots\ldots (18)$$

which is the required exact solution of (15).

Application: 4 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u''(x) = x + \int_0^x (x-t)u(t)dt \\ with\, u(0) = 0, u'(0) = 1 \end{array}\right\} \ldots\ldots (19)$$

Applying the Mahgoub transform to both sides of (19) and using initial condition, we have

$$v^2 M\{u(x)\} = v + \frac{1}{v} + M\left\{\int_0^x (x-t)\,u(t)dt\right\} \ldots (20)$$

Using convolution theorem of Mahgoub transform on (20) and simplify, we have

$$M\{u(x)\} = \frac{v}{v^2-1} \ldots \ldots (21)$$

Operating inverse Mahgoub transform on both sides of(21), we have

$$u(x) = M^{-1}\left\{\frac{v}{v^2-1}\right\} = sinhx \ldots\ldots (22)$$

which is the required exact solution of (19).

X. CONCLUSION

In this paper, we have successfully developed the Mahgoub transform for solving linear Volterra integro-differential equation of second kind. The given applications show that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other linear Volterra integral equations and their system.

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AUTHORS

Sudhanshu Aggarwal received his M.Sc. degree from M.S. College, Saharanpur in 2007. He has also qualified CSIR NET examination (June-2010, June-2012, June-2013, June-2014 & June-2015) in Mathematical Sciences. He is working as an Assistant Professor in National P.G. College Barhalganj Gorakhpur. He is equipped with an extraordinary caliber and appreciable academic potency. He has around ten years of teaching experience at various engineering colleges affiliated to AKTU. His fields of interest include Integral Transform Methods, Differential and Partial Differential Equations, Integral Equations and Number Theory. He has published many research papers in national and international journals.

Nidhi Sharma received her M.Sc. degree from Banasthali University Rajasthan in 2015. She is working as an Assistant Professor in Noida Institute of Engineering & Technology, Greater Noida since 2015. She has also qualified IIT Jam exam (2013).She has published many research papers in national and international journals. Her fields of interest include Optimization, Fuzzy Logic and Topological Spaces.

Raman Chauhan has completed MSc. (Mathematics) from Indian Institute of Technology, Roorkee in 2011.He is working at NIET, Greater Noida since 2013 as a dynamic, successful and responsible faculty member in the department of Mathematics. He has also qualified IIT Jam exam (2009) and Gate exam (2011, 2013, 2014 & 2015).His fields of interest include Optimization, Integral Equations and Abstract Algebra.