Solution of the Homogeneous and Non Homogeneous Diffusion Heat Equation by Laplace Transform of Homotopy Perturbation Method

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Abstract: **In this paper, I apply a combined form of the laplace transform method with the homotopy perturbation method (HPM) to obtain the solution of Diffusion equation. This method is called laplace transform of homotopy pertubation method (LTHPM). LTHPM is applied to solve a linear and nonlinear homogeneous problems.**

Keywords: **Diffusion Equation, Laplace Transform, HPM, LTHPM.**

I. INTRODUCTION

The advection-Diffusion equations have found wide applications in the field of applied mathematics, soil mechanics and plenty of other engineering area. In this paper LTHPM is used to clear up the advection-Diffusion equations.

The homotopy perturbation method (HPM) proposed by the Chinese mathematician Ji-Huan He[1-4] of Shanghai University in 1998 and additionally enhanced by D.D.Ganji [5]. These days many Researchers used Laplace transform with homotopy perturbation method. He Laplace Method for linear and Nonlinear Partial differential equations by Hradyesh Kumar Mishra and Atulya K.Nagar [6]. Homotopy perturbation algorithm using Laplace transform for dynamic equation by Jagdev Singh, Devendra Kumar[7]. LTHPM is very easy for solving linear and nonlinear problems.

II. BASIC IDEA OF LAPLACE TRANSFORM OF HOMOTOPY PERTURBATION METHOD

Consider a non linear Partial Differential Equations (PDE) with the initial conditions

$$Du(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \qquad (1)$$

$$u(x,0)=h(x),u_t(x,0)=f(x)$$

where $D=$ second order linear differential operator = $\frac{\partial^2}{\partial t^2}$

$R=$ linear differential operator of less order than D

N =general nonlinear differential operator

$g(x,t)=$ source term

Taking Laplace transform on (1) both sides, we get

$$L\left[Du(x,t)\right]+L\left[Ru(x,t)\right]+L\left[Nu(x,t)\right]=L\left[g(x,t)\right] \qquad (2)$$

Using the Laplace transform of differentiation with boundary conditions, we get

$$L\left[u(x,t)\right]=\frac{h(x)}{s}+\frac{f(x)}{s^2}+\frac{1}{s^2}L\left[g(x,t)\right]-\frac{1}{s^2}L\left[Ru(x,t)\right]-\frac{1}{s^2}L\left[Nu(x,t)\right]$$

OR

$$u(x,t)=G(x,t)-L^{-1}\left[\frac{1}{s^2}L\left[Ru(x,t)\right]+\frac{1}{s^2}L\left[Nu(x,t)\right]\right] \qquad (3)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the HPM

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t) \qquad (4)$$

and the nonlinear term can be decomposed as

$$Nu(x,t)=\sum_{n=0}^{\infty}p^n H_n(u) \qquad (5)$$

for some He's polynomials $H_n(u)$ that are given by

$$H_n(u_0,u_1,u_2,.....u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right], n=0,1,2,3,...... \qquad (6)$$

substituting equations (4) and (5) in (3) we get

$$\sum_{n=0}^{\infty} p^n u(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{7}$$

which is the coupling of the Laplace transform and the HPM using He's polynomials. Comparing the coefficient of like powers of p the following approximations are obtained

$$p^0 : u_0(x,t) = G(x,t),$$
$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[Ru_0(x,t) + H_0(u)\right]\right]$$
$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[Ru_1(x,t) + H_1(u)\right]\right]$$
$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[Ru_2(x,t) + H_2(u)\right]\right] \tag{8}$$

....

The best approximation for the solution is

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \tag{9}$$

III. LINEAR HOMOGENEOUS DIFFUSION EQUATION

Consider linear homogeneous Diffusion Equation

$$u_t = u_{xx} - u, 0 < x < 1, t > 0 \tag{10}$$

Boundary conditions are given by

$$u(0,t) = 0 = u(1,t), t > 0 \tag{11}$$

Initial condition is given by

$$u(x,0) = \sin \pi x, 0 < x < 1 \tag{12}$$

This is a Heat Equation which is solved by LTHPM.

Taking Laplace Transform on both sides of Equation (10) subject to the initial conditions

$$u(x,s) = \frac{\sin \pi x}{s} - \frac{1}{s}L\left[u - u_{xx}\right] \tag{13}$$

Taking Inverse Laplace Transform on both sides, we get

$$u(x,t) = \sin \pi x - L^{-1}\left[\frac{1}{s}L\left[u - u_{xx}\right]\right]$$

Now, we apply the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sin \pi x - p\left(L^{-1}\left[\frac{1}{s}L\left[u - u_{xx}\right]\right]\right) \tag{14}$$

Comparing the powers of p, we have

$$p^0 : u_0(x,t) = \sin \pi x$$
$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s}L\left(u_0 - u_{0xx}\right)\right] = -\sin \pi x\left(\pi^2 + 1\right)t$$
$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s}L\left(u_1 - u_{1xx}\right)\right] = \sin \pi x\left(\pi^2 + 1\right)^2 \frac{t^2}{2!}$$
$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s}L\left(u_2 - u_{2xx}\right)\right] = -\sin \pi x\left(\pi^2 + 1\right)^3 \frac{t^3}{3!}$$

....

Substituting this value in equation (9), we get

$$u(x,t) = \sin \pi x - \sin \pi x\left(\pi^2+1\right)t + \sin \pi x\left(\pi^2+1\right)^2 \frac{t^2}{2!} - \sin \pi x\left(\pi^2+1\right)^3 \frac{t^3}{3!} + ... \tag{15}$$

$$= \sin \pi x\left[1 - \left(\pi^2+1\right)t + \left(\pi^2+1\right)^2 \frac{t^2}{2!} - \left(\pi^2+1\right)^3 \frac{t^3}{3!} + ...\right]$$

$$= \sin \pi x e^{-\left(\pi^2+1\right)t} \tag{16}$$

This is the exact solution of (10).The solution of the problem is similar to the Variational homotopy perturbation method [8].

IV. NONLINEAR HOMOGENEOUS DIFFUSION EQUATION

Consider nonlinear homogenous Diffusion Equation

$$u_t - 3u_{xx} = x, 0 < x < \pi, t > 0 \tag{17}$$

Boundary conditions are given by

$$u(0,t) = 0, u(\pi,t) = \pi t \tag{18}$$

Initial condition is given by

$$u(x,0) = \sin x, 0 < x < \pi \tag{19}$$

This is a Heat equation which is solved by LTHPM

Taking Laplace Transform on both sides of Equation (17) subject to the initial conditions

$$u(x,s) = \frac{\sin x}{s} - \frac{1}{s} L\left[-3u_{xx} - x\right] \tag{20}$$

Taking Inverse Laplace Transform on both sides, we get

$$u(x,t) = \sin x - L^{-1}\left[\frac{1}{s} L\left[-3u_{xx} - x\right]\right]$$

Now, we apply the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sin x - p\left(L^{-1}\left[\frac{1}{s} L\left[-3u_{xx} - x\right]\right]\right) \tag{21}$$

Comparing the powers of p, we have

$$p^0 : u_0(x,t) = \sin x$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s} L\left(-3u_{0xx} - x\right)\right] = -(3\sin x - x)t = -3t\sin x + xt$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s} L\left(-3u_{1xx}\right)\right] = \frac{9t^2}{2!}\sin x$$

$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s} L\left(-3u_{2xx}\right)\right] = -\frac{27t^3}{3!}\sin x$$

....

Substituting this value in equation (9), we get

$$u(x,t) = \sin x - 3t\sin x + xt + \frac{9t^2}{2!}\sin x - \frac{27t^3}{3!}\sin x + ...$$

$$= \sin x\left[1 - 3t + \frac{9t^2}{2!} - \frac{27t^3}{3!} + ...\right] + xt$$

$$= \sin x e^{-3t} + xt$$

This is the exact solution of (17).The solution of the problem is similar to the Homotopy Perturbation Method [9].

V. CONCLUSIONS

In this paper, the Laplace Transform of Homotopy Perturbation Method has effectively used for solving Diffusion equations. The outcomes show that LTHPM is powerful and really green approach in finding the analytical solutions for a Differential equations. This method is decreasing the computational work and maintaining the high accuracy.

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