Solution of Linear Partial Integro-Differential Equations using Kamal Transform

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Abstract: **In this paper, we used Kamal transform for solving linear partial integro-differential equations. The technique is described and illustrated with application. This technique gives the exact results using very less computational work.**

Keywords: **Linear partial integro-differential equation, Kamal transform, Convolution theorem, Inverse Kamal transform.**

I. INTRODUCTION

Mathematical modeling of real life problems usually results in functional equations e.g. differential equations, partial differential equations, integral equations, integro-differential equations, stochastic equations, delay differential equations, partial integro-differential equations and others. In particular partial integro-differential equations arise in many scientific and engineering applications such as mathematical physics, visco-elasticity, finance, heat transfer, diffusion process, nuclear reactor dynamics, in general neutron diffusion, nano-hydrodynamics and fluid dynamics.

The general linear partial integro-differential equation is given by

$$\sum_{i=0}^{m} a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i\,(t,s) \frac{\partial^i u(x,s)}{\partial x^i} + f(x,t) = 0 \ldots\ldots (1)$$

(with prescribed conditions), where the kernels $k_i(t,s)$ and $f(x,t)$ are known functions and a_i, b_i, c and d_i are constants or functions of x.

The Kamal transform of the function $F(t)$is defined as [6]:

$$K\{F(t)\} = \int_0^\infty F(t) e^{\frac{-t}{v}} dt$$
$$= G(v), t \geq 0, k_1 \leq v \leq k_2 \ldots\ldots (2)$$

Where K is Kamal transform operator.

The Kamal transform of the function $F(t)$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Kamal tansform of the function $F(t)$.

Appell et al. [1] discussed the partial integral operators and integro differential equations. Bahuguna and Dabas [2] gave existence and uniqueness of a solution to a PIDE by the method of lines. Yanik and Fairweather [3] used finite element methods for parabolic and hyperbolic partial integro-differential equations. Dehghan [4] discussed the solution of a partial integro-differential equation arising from visco elasticity. Efficient solution of a partial integro-differential equation in finance was given by Sachs and Strauss [5]. Abdelilah and Hassan [6] gave a new integral transform “Kamal Transform”. Abdelilah and Hassan [7] applied Kamal transform for solving partial differential equations. Fadhil [8] gave the convolution for Kamal and Mahgoub transforms. Taha et al. [9] discussed the dualities between Kamal and Mahgoub integral transforms and also gave some famous integral transforms. For modeling biofluids flow in fractured biomaterials, Zadeh [10] gave an integro-partial differential equation. Thorwe and Bhalekar [11] used Laplace transform method for solving partial integro-differential equations. Mohand and Tarig [12] applied Elzaki transform method for solving partial integro-differential equations. Aboodh et al. [13] gave the solution of partial integro-differential equations by using Aboodh and double Aboodh transform methods. Mohand [15] used double Elzaki transform method for solving partial integro-differential equations. Aggarwal et al. [16] discussed a new application of Kamal transform for solving linear Volterra integral equations.

The object of the present study is to determine exact solutions for linear partial integro-differential equations using Mahgoub transform without large computational work.

II. LINEARITY PROPERTY OF KAMAL TRANSFORMS

$$K\{aF(t) + bG(t)\} = aK\{F(t)\} + bK\{G(t)\}$$

Where a, b are arbitrary constants.

III. KAMAL TRANSFORM OF SOME ELEMENTARY FUNCTIONS [6, 8]:

S.N.	$F(t)$	$K\{F(t)\} = G(v)$
1.	1	v
2.	t	v^2
3.	t^2	$2!\,v^3$
4.	$t^n, n \in N$	$n!\,v^{n+1}$
5.	e^{at}	$\frac{v}{1-av}$
6.	$sinat$	$\frac{av^2}{1+a^2v^2}$
7.	$cosat$	$\frac{v}{1+a^2v^2}$
8.	$sinhat$	$\frac{av^2}{1-a^2v^2}$
9.	$coshat$	$\frac{v}{1-a^2v^2}$

IV. KAMAL TRANSFORM OF SOME PARTIAL DERIVATIVES OF THE FUNCTION $u(x,t)$[7]:

If $K\{u(x,t)\} = G(x,v)$ then

(a) $K\left\{\frac{\partial u(x,t)}{\partial t}\right\} = \frac{1}{v}G(x,v) - u(x,0) \ldots\ldots(3)$

(b) $K\{\frac{\partial^2 u(x,t)}{\partial t^2}\} = \frac{1}{v^2}G(x,v) - \frac{1}{v}u(x,0)$
$-u_t(x,0) \ldots\ldots(4)$

(c) $K\left\{\frac{\partial^n u(x,t)}{\partial t^n}\right\} = \frac{1}{v^n}G(x,v) - \frac{1}{v^{n-1}}u(x,0)$

$$-\frac{1}{v^{n-2}}u_t(x,0) - \cdots \ldots - u_{ttt\ldots(n-1)times}(x,0) \ldots .(5)$$

(d) $K\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{dG(x,v)}{dx} \ldots\ldots\ldots(6)$

(e) $K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{d^2G(x,v)}{dx^2} \ldots\ldots..(7)$

(f) $K\left\{\frac{\partial^n u(x,t)}{\partial x^n}\right\} = \frac{d^nG(x,v)}{dx^n} \ldots\ldots..(8)$

V. CONVOLUTION OF TWO FUNCTIONS [14]:

Convolution of two functions $F(t)$ and $H(t)$ is denoted by $F(t) * H(t)$ and it is defined by

$$F(t) * H(t) = F * H = \int_0^t F(x)H(t-x)dx$$
$$= \int_0^t H(x)F(t-x)dx$$

VI. CONVOLUTION THEOREM FOR KAMAL TRANSFORMS [8]:

If $K\{F(t)\} = G(v)$ and $K\{H(t)\} = I(v)$ then

$$K\{F(t) * H(t)\} = K\{F(t)\}K\{H(t)\} = G(v)I(v)$$

VII. INVERSE KAMAL TRANSFORMS

If $K\{F(t)\} = G(v)$ then $F(t)$ is called the inverse Kamal transform of $G(v)$ and mathematically it is defined as

$F(t) = K^{-1}\{G(v)\}$

Where K^{-1} is the inverse Kamal transform operator.

VIII. INVERSE KAMAL TRANSFORM OF SOME ELEMENTARY FUNCTIONS

S.N.	$G(v)$	$F(t) = K^{-1}\{G(v)\}$
1.	v	1
2.	v^2	t
3.	v^3	$\frac{t^2}{2!}$
4.	$v^{n+1}, n \in N$	$\frac{t^n}{n!}$
5.	$\frac{v}{1-av}$	e^{at}
6.	$\frac{v^2}{1+a^2v^2}$	$\frac{sinat}{a}$
7.	$\frac{v}{1+a^2v^2}$	$cosat$
8.	$\frac{v^2}{1-a^2v^2}$	$\frac{sinhat}{a}$
9.	$\frac{v}{1-a^2v^2}$	$coshat$

IX. KAMAL TRANSFORM FOR LINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS:

In this section, we present Kamal transform for solving linear partial integro-differential equations given by (1). In this work, we will assume that the kernels $k_i(t,s)$ of (1) are difference kernel that can be expressed by difference $(t-s)$. The linear partial integro-differential equation (1) can thus be expressed as

$$\sum_{i=0}^{m} a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i\,(t-s)\frac{\partial^i u(x,s)}{\partial x^i} + f(x,t) = 0 \ldots\ldots(9)$$

Applying the Kamal transform to both sides of(9), we have

$$\sum_{i=0}^{m} a_i K\left\{\frac{\partial^i u(x,t)}{\partial t^i}\right\} + \sum_{i=0}^{n} b_i K\left\{\frac{\partial^i u(x,t)}{\partial x^i}\right\} + cK\{u\} + \sum_{i=0}^{r} d_i\, K\left\{\int_0^t k_i\,(t-s)\,\frac{\partial^i u(x,s)}{\partial x^i}\right\} + K\{f(x,t)\} = 0 \ldots\ldots\ldots (10)$$

Using convolution theorem of Kamal transform and equations (5) and (8) in equation (10), we have

$$\sum_{i=0}^{m} a_i \left[\begin{matrix}\frac{1}{v^i}G(x,v) - \frac{1}{v^{i-1}}u(x,0) - \frac{1}{v^{i-2}}u_t(x,0) - \\ \ldots\ldots - u_{ttt\,\ldots.(i-1)times}\,(x,0)\end{matrix}\right]$$

$$+\sum_{i=0}^{n} b_i \frac{d^i G(x,v)}{dx^i} + cG(x,v)$$

$$+\sum_{i=0}^{r} d_i\, \bar{k_i}(v)\frac{d^i G(x,v)}{dx^i} + \bar{f}(x,v) = 0 \ldots..(11)$$

where $K\{u(x,t)\} = G(x,v)$, $K\{k_i(t)\} = \bar{k_i}(v)$ and $K\{f(x,t)\} = \bar{f}(x,v)$.

After using prescribed conditions, equation (11) represents an ordinary differential equation with dependent variable $G(x,v)$. After solving this ordinary differential equation and taking inverse Kamal transform of $G(x,v)$, we have the required solution $u(x,t)$ of equation (1).

X. APPLICATION

In this section, an application is given in order to demonstrate the effectiveness of Kamal transform for solving linear partial integro-differential equation.

Consider the linear partial integro-differential equation [11-13]

$$u_{tt} = u_x + 2\int_0^t (t-s)\,u(x,s)ds - 2e^x \ldots(12)$$

with initial conditions

$$u(x,0) = e^x, u_t(x,0) = 0 \ldots\ldots(13)$$

and boundary condition

$$u(0,t) = cost \ldots\ldots\ldots\ldots\ldots\ldots.(14)$$

Applying the Kamal transform to both sides of equation (12), we have

$$K\{u_{tt}\} = K\{u_x\} + 2K\left\{\int_0^t (t-s)\,u(x,s)ds\right\} - 2e^x K\{1\} \ldots\ldots\ldots\ldots(15)$$

Using convolution theorem of Kamal transform and equations (4) and (6) in equation (15), we have

$$\frac{1}{v^2}G(x,v) - \frac{1}{v}u(x,0) - u_t(x,0)$$

$$= \frac{dG(x,v)}{dx} + 2v^2 G(x,v) - 2ve^x \ldots\ldots\ldots(16)$$

Now using equation(13) in equation (16), we have

$$\frac{dG(x,v)}{dx} + \left(2v^2 - \frac{1}{v^2}\right)G(x,v) = \left(2v - \frac{1}{v}\right)e^x \ldots\ldots.(17)$$

which is an ordinary linear differential equation.

The general solution of equation (17) is give by

$$G(x,v) = e^x\left(\frac{v}{1+v^2}\right) + ce^{-\left(2v^2 - \frac{1}{v^2}\right)x} \ldots.(18)$$

Now, using equation (14), we have

$$K\{u(0,t)\} = G(0,v) = K\{cost\} = \frac{v}{1+v^2} \ldots.(19)$$

Using equation (19) and equation (18), we have

$$c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Substituting the value of c from equation (20) into equation (18), we have

$$G(x,v) = e^x\left(\frac{v}{1+v^2}\right)\ldots\ldots\ldots\ldots.(21)$$

Operating inverse Kamal transform on both sides of equation (21), we have

$$u(x,t) = K^{-1}\{G(x,v)\} = e^x K^{-1}\left\{\frac{v}{1+v^2}\right\} = e^x cost \ldots\ldots\ldots\ldots..(22)$$

Which is the required exact solution of equation (12) with equations (13) and (14).

XI. CONCLUSION

In this paper, we have successfully developed the Kamal transform for solving linear partial integro-differential equation. The given application shows that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other linear partial integro-differential equations.

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