Application of Laplace Transform for Solving Population Growth and Decay Problems

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Abstract: **The population growth and decay problems arise in the field of chemistry, physics, biology, social science, zoology etc. In this paper, we used Laplace transform for solving population growth and decay problems and some applications are given in order to demonstrate the effectiveness of Laplace transform for solving population growth and decay problems.**

Keywords: **Laplace transform, Inverse Laplace transform, Population growth problem, Decay problem, Half-life.**

I. INTRODUCTION

The population growth (growth of a plant, or a cell, or an organ, or a species) is governed by the first order linear ordinary differential equation [1-10]

$$\frac{dN}{dt} = KN \dots\dots\dots\dots\dots (1)$$

with initial condition as

$$N(t_0) = N_0 \dots\dots\dots\dots (2)$$

Where K is a positive real number, N is the amount of population at time t and N_0 is the initial population at time t_0.

Equation (1) is known as the Malthusian law of population growth.

Mathematically the decay problem of the substance is defined by the first order linear ordinary differential equation [7, 9-10]

$$\frac{dN}{dt} = -KN \dots\dots\dots\dots\dots (3)$$

with initial condition as

$$N(t_0) = N_0 \dots\dots\dots\dots (4)$$

where N is the amount of substance at time t, K is a positive real number and N_0 is the initial amount of the substance at time t_0.

In equation (3), the negative sign in the R.H.S. is taken because the mass of the substance is decreasing with time and so the derivative $\frac{dN}{dt}$ must be negative.

The Laplace transform of the function $F(t)$ for all $t \geq 0$ is defined as [11-15]:

$$L\{F(t)\} = \int_0^\infty F(t)e^{-st}dt = f(s) \dots. (6)$$

where L is Laplace transform operator.

The Laplace transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Laplace transform of the function $F(t)$.

The aim of this work is to finding the solution of population growth and decay problems using Laplace transform without large computational work.

II. LINEARITY PROPERTY OF LAPLACE TRANSFORM [12]

If $L\{F(t)\} = f(s)$ and $L\{G(t)\} = g(s)$ then

$$L\{aF(t) + bG(t)\} = aL\{F(t)\} + bL\{G(t)\}$$

$$\Rightarrow L\{aF(t) + bG(t)\} = af(s) + bg(s),$$

where a, b are arbitrary constants.

III. LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS [11-1 5]

S.N.	$F(t)$	$L\{F(t)\} = f(s)$
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	t^2	$\frac{2!}{s^3}$
4.	$t^n, n \in N$	$\frac{n!}{s^{n+1}}$

5.	$t^n, n > -1$	$\frac{\Gamma(n+1)}{s^{n+1}}$
6.	e^{at}	$\frac{1}{s-a}$
7.	$sinat$	$\frac{a}{s^2+a^2}$
8.	$cosat$	$\frac{s}{s^2+a^2}$
9.	$sinhat$	$\frac{a}{s^2-a^2}$
10.	$coshat$	$\frac{s}{s^2-a^2}$

IV. INVERSE LAPLACE TRANSFORM [11-12]

If $L\{F(t)\} = f(s)$ then $F(t)$ is called the inverse Laplace transform of $f(s)$ and mathematically it is defined as

$F(t) = L^{-1}\{f(s)\}$

where L^{-1} is the inverse Laplace transform operator.

V. INVERSE LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS [11-13]

S.N.	$f(s)$	$F(t) = L^{-1}\{f(s)\}$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^3}$	$\frac{t^2}{2!}$
4.	$\frac{1}{s^{n+1}}, n \in N$	$\frac{t^n}{n!}$
5.	$\frac{1}{s^{n+1}}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{1}{s-a}$	e^{at}
7.	$\frac{1}{s^2+a^2}$	$\frac{sinat}{a}$
8.	$\frac{s}{s^2+a^2}$	$cosat$
9.	$\frac{1}{s^2-a^2}$	$\frac{sinhat}{a}$
10.	$\frac{s}{s^2-a^2}$	$coshat$

VI. LAPLACE TRANSFORM OF DERIVATIVES OF THE FUNCTION $F(t)$ [11-13]

If $L\{F(t)\} = f(s)$ then

a) $L\{F'(t)\} = sf(s) - F(0)$
b) $L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$
c) $L\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots \ldots \ldots - F^{(n-1)}(0)$

VII. LAPLACE TRANSFORM FOR POPULATION GROWTH PROBLEM

In this section, we present Laplace transform for population growth problem given by (1) and (2).

Applying the Laplace transform on both sides of (1), we have

$L\left\{\frac{dN}{dt}\right\} = KL\{N(t)\} \ldots \ldots \ldots \ldots \ldots \ldots .(5)$

Now applying the property, Laplace transform of derivative of function, on (5), we have

$sL\{N(t)\} - N(0) = KL\{N(t)\} \ldots \ldots (6)$

Using (2) in (6) and on simplification, we have

$(s-K)\, L\{N(t)\} = N_0$

$\Rightarrow L\{N(t)\} = \frac{N_0}{(s-K)} \ldots \ldots \ldots \ldots \ldots (7)$

Operating inverse Laplace transform on both sides of (7), we have

$N(t) = L^{-1}\left\{\frac{N_0}{(s-K)}\right\}$

$\Rightarrow N(t) = N_0 L^{-1}\left\{\frac{1}{(s-K)}\right\}$

$\Rightarrow N(t) = N_0 e^{Kt} \ldots \ldots \ldots \ldots \ldots \ldots \ldots (8)$

which is the required amount of the population at time t.

VIII. LAPLACE TRANSFORM FOR DECAY PROBLEM

In this section, we present Laplace transform for decay problem which is mathematically given by (3) and (4).

Applying the Laplace transform on both sides of (3), we have

$L\left\{\frac{dN}{dt}\right\} = -KL\{N(t)\} \ldots \ldots \ldots \ldots \ldots \ldots .(9)$

Now applying the property, Laplace transform of derivative of function, on (9), we have

$sL\{N(t)\} - N(0) = -KL\{N(t)\} \ldots (10)$

Using (4) in (10) and on simplification, we have

$(s+K)\, L\{N(t)\} = N_0$

$$\Rightarrow L\{N(t)\} = \frac{N_0}{(s+K)} \dots\dots\dots\dots\dots (11)$$

Operating inverse Laplace transform on both sides of (11), we have

$$N(t) = L^{-1}\left\{\frac{N_0}{(s+K)}\right\}$$

$$\Rightarrow N(t) = N_0 L^{-1}\left\{\frac{1}{(s+K)}\right\}$$

$$\Rightarrow N(t) = N_0 e^{-Kt} \dots\dots\dots\dots\dots (12)$$

which is the required amount of substance at time t.

IX. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Laplace transform for solving population growth and decay problems.

Application: 1 The population of a city grows at a rate proportional to the number of people presently living in the city. If after two years, the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the city.

This problem can be written in mathematical form as:

$$\frac{dN(t)}{dt} = KN(t) \dots\dots\dots\dots\dots (13)$$

where N denote the number of people living in the city at any time t and K is the constant of proportionality. Consider N_0 is the number of people initially living in the city at $t = 0$.

Taking the Laplace transform on both sides of (13), we have

$$L\left\{\frac{dN}{dt}\right\} = KL\{N(t)\} \dots\dots\dots\dots\dots\dots (14)$$

Now applying the property, Laplace transform of derivative of function, on (14), we have

$$sL\{N(t)\} - N(0) = KL\{N(t)\} \dots\dots (15)$$

Since at $t = 0, N = N_0$, so using this in (15), we have

$$(s-K)\, L\{N(t)\} = N_0$$

$$\Rightarrow L\{N(t)\} = \frac{N_0}{(s-K)} \dots\dots\dots\dots (16)$$

Operating inverse Laplace transform on both sides of (16), we have

$$N(t) = L^{-1}\left\{\frac{N_0}{(s-K)}\right\}$$

$$\Rightarrow N(t) = N_0 L^{-1}\left\{\frac{1}{(s-K)}\right\}$$

$$\Rightarrow N(t) = N_0 e^{Kt} \dots\dots\dots\dots\dots\dots\dots (17)$$

Now at $= 2$, $N = 2N_0$, so using this in (17), we have

$$2N_0 = N_0 e^{2K}$$

$$\Rightarrow e^{2K} = 2$$

$$\Rightarrow K = \frac{1}{2} log_e 2 = 0.347 \dots\dots\dots\dots\dots (18)$$

Now using the condition at $= 3$, $N = 20{,}000$, in (17), we have

$$20{,}000 = N_0 e^{3K} \dots\dots\dots\dots (19)$$

Putting the value of K from (18) in (19), we have

$$20{,}000 = N_0 e^{3\times 0.347}$$

$$\Rightarrow 20{,}000 = 2.832 N_0$$

$$\Rightarrow N_0 \simeq 7062 \dots\dots\dots\dots\dots\dots (20)$$

which are the required number of people initially living in the city.

Application: 2 A radioactive substance is known to decay at a rate proportional to the amount present. If initially there is 100 milligrams of the radioactive substance present and after two hours it is observed that the radioactive substance has lost 10 percent of its original mass, find the half life of the radioactive substance.

This problem can be written in mathematical form as:

$$\frac{dN(t)}{dt} = -KN(t) \dots\dots\dots\dots\dots (21)$$

where N denote the amount of radioactive substance at time t and K is the constant of proportionality. Consider N_0 is the initial amount of the radioactive substance at time $t = 0$.

Applying the Laplace transform on both sides of (21), we have

$$L\left\{\frac{dN}{dt}\right\} = -KL\{N(t)\} \dots\dots\dots\dots\dots\dots (22)$$

Now applying the property, Laplace transform of derivative of function, on (22), we have

$$sL\{N(t)\} - N(0) = -KL\{N(t)\} \dots (23)$$

Since at $t = 0, N = N_0 = 100$, so using this in (23), we have

$$sL\{N(t)\} - 100 = -KL\{N(t)\}$$

$$\Rightarrow (s+K)\, L\{N(t)\} = 100$$

$$\Rightarrow L\{N(t)\} = \frac{100}{(s+K)} \dots\dots\dots\dots (24)$$

Operating inverse Laplace transform on both sides of (24), we have

$$N(t) = L^{-1}\left\{\frac{100}{(s+K)}\right\}$$

$$= 100L^{-1}\left\{\frac{1}{(s+K)}\right\}$$

$$\Rightarrow N(t) = 100e^{-Kt} \dots \dots ..(25)$$

Now at $t = 2$, the radioactive substance has lost 10 percent of its original mass 100 mg so $N = 100 - 10 = 90$, using this in (25), we have

$$90 = 100e^{-2K}$$

$$\Rightarrow e^{-2K} = 0.90$$

$$\Rightarrow K = -\frac{1}{2}log_e 0.90 = 0.05268 \dots \dots \dots (26)$$

We required t when $N = \frac{N_0}{2} = \frac{100}{2} = 50$ so from (25), we have

$$50 = 100e^{-Kt} \dots \dots \dots \dots ..(27)$$

Putting the value of K from (26) in (27), we have

$$50 = 100e^{-0.05268\,t}$$

$$\Rightarrow e^{-0.05268\,t} = 0.50$$

$$\Rightarrow t = -\frac{1}{0.05268}log_e 0.50$$

$$\Rightarrow t = 13.157 hours \dots \dots \dots ...(28)$$

which is the required half-time of the radioactive substance.

X. CONCLUSION

In this paper, we have successfully developed the Laplace transform for solving the population growth and decay problems. The given applications show that the effectiveness of Laplace transform for solving population growth and decay problems. The proposed scheme can be applied for the continuous compound interest and heat conduction problems.

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