Variable Step - Size and The Effect on Accuracy of an Implicit One – Step Multiderivative Method for Solving Non – Stiff and Stiff First Order O.D.Es.

Oludolapo Kehinde Famurewa[1*], Roseline Bosede Ogunrinde[2], Remi Julius Dare[1]
[1] Department of Mathematical Sciences, Kings University, Odeomu, Nigeria
[2] Ekiti State University, Ado – Ekiti Nigeria

***Abstract:* - This articles presents variation effect of step – size on the accuracy pattern of an implicit one - step multiderivative method of the form:**

$$\sum_{j=0}^{1}\alpha_j\, y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{1}\beta_{ij}\, y^i{}_{n+j}, \quad \alpha_k = +1$$

which is a variant of the Implicit Multiderivative linear multistep method of the form:

$$\sum_{j=0}^{k}\alpha_j\, y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k}\beta_{ij}\, y^i{}_{n+j}, \quad \alpha_k = +1$$

(where *h* is the step size, *k* is the step number, *l* is the order of the derivative, α_j and β_j are unknown parameters to be determined).

Step size (*h*) of 0.1, 0.01 and 0.001 were used for the implementation of the method to solven on – stiff and stiff initial value problem of first order Ordinary Differential Equation. Accuracy pattern of the method when it was used to solve a non – stiff problem showed that when h=0.01 the computed result was more accurate than when h=0.1 and h=0.001. When the method was used to solve a stiff initial value problem, accuracy improved as the step size (*h*) reduced. This study therefore confirmed the assertion that reduction in step – size improves accuracy of numerical methods and further showed that for optimal accuracy to be achieved in non - stiff problems, there is limitation to reduction in step size, because h=0.01 gave better result in terms of accuracy than h=0.001.

***Keywords:* Step – size, Accuracy, Implicit, Multi-derivative, One-step, Ordinary Differential Equation, Non – stiff, Stiff**

I. INTRODUCTION

The aim of numerical methods is to provide practical procedure for calculating the solutions of problems in applied mathematics to a specified degree of accuracy. These practical procedures can be achieved through approximations to the given equation; for this, it is necessary to ensure that the results are correct to within stated limits. Accuracy is therefore one of the major conditions to be satisfied by a good numerical method and it is determined by how close to the exact solution a numerically computed solution is. Butcher (2008) states that a numerical method is accurate if and only if its result tends to the exact solution. According to Leveque(2006) and Strikwerda (2004); in numerical analysis, order of accuracy quantifies the rate of convergence of a numerical approximation of a differential equation to the exact solution. A good numerical method for solution of non-stiff and stiff initial value problems is therefore required to be accurate, consistent, zero-stable and convergent while an additional condition for stiff initial value problem is absolute stability [Lambert (1973)].

Since numerical methods usually give approximate solutions which consist of generating some sequence of approximations to the solutions which hopefully is expected to converge to the exact solution, there is expedient need to work on the existing methods in order to improve their accuracy, efficiency and effectiveness.

According to Famurewa (2011), an attempt to improve upon the accuracy of the conventional Linear Multistep Method(LMM) of the form:

$$\sum_{j=0}^{k}\alpha_j\, y_{n+j} = h\sum_{j=0}^{k}\beta_j\, y^1_{n+j}, \quad \alpha_k = +1 \qquad \text{...... (1)}$$

led to the derivation of the Implicit Multiderivative Linear Multistep Method (IMLMM) of the form:

$$\sum_{j=0}^{k}\alpha_j\, y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k}\beta_{ij}\, y^i{}_{n+j}, \quad \alpha_k = +1 \qquad \text{.......... (2)}$$

with local truncation error of the form:

$$T_{n+1} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} \, y^i{}_{n+j} - \sum_{j=0}^{k} \alpha_j \, y_{n+j} \quad , \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

(where h is the step - size, k is the step number, l is the order of the derivative, α_j and β_{ij} are unknown parameters to be determined).

Equation (2) was derived by introducing more derivative properties into equation (1) to make it more accurately suitable for the numerical solution of theoretical and practical problems involving ordinary differential equations emanating from various disciplines.

Varying the order of derivative (l) while keeping the step number (k) in equation (2) constant at k=1resulted into the derivation of one step multiderivative method of the form:

$$\sum_{j=0}^{1} \alpha_j \, y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{1} \beta_{ij} \, y^i{}_{n+j} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

with local truncation error of the form:

$$T_{n+1} = \sum_{i=1}^{l} h^i \sum_{j=0}^{1} \beta_{ij} \, y^i{}_{n+j} - \sum_{j=0}^{1} \alpha_j \, y_{n+j} \quad , \qquad \ldots\ldots\ldots \ldots(5)$$

Although, according to Ezzeddine and Hojjatti (2012), the existence and computation of high derivatives put a restriction on multiderivative methods, however, for a one-step method, the computational cost and time are considerable when compared with some other methods. It was discovered in Famurewa (2011) that multiderivative method is superior when high accuracy is requested for non - stiff and stiff initial value problems of first order ordinary differential equation (O.D.E.).

In Famurewa and Olorunsola (2013), one step Multiderivative method was investigated and found to be accurate, consistent, zero stable, convergent and absolutely stable with optimal accuracy at *l=3* when the method was implemented using stiff and non – stiff initial value problem (IVP) of first order O. D. E. with step size (h) =0.1

According to Majid (2010), step size is one of the factors that influence numerical accuracy as well as the computation time, this informed this study which is focused at determining the accuracy pattern of the method when the step size (h) is reduced, hence the study aimed at investigating the accuracy pattern of the one step Multiderivative method at *h=0.1*, *h=0.01* and *h=0.001*.

II. REASERCH METHODOLOGY

The one step multiderivative method was transformed into computer algorithm using MATLAB program and was used to solve both non – stiff and stiff IVP of first order O.D.E. at three (3) different values of step size:h =0.1,h = 0.01 and h = 0.001 respectively.

Table of results as well as table of errors were computed to determine the step size that yielded the most accurate result.

III. IMPLIMENTATION

To access the accuracy and convergence of the method, the program was implemented on a digital computer using:

(1) A non - stiff initial value problem

$$y^1 = x + y, y(0) = 1, x \in [0,1]$$

Exact solution:$y(x) = 2e^x - x - 1$

(2) A stiff initial value problem

$$y^1 = -10(y - x^3) + 3x^2, y(0) = 1, x \in [0,1]$$

Exact solution:$y(x) = x^3 + e^{-10x}$

$at h = 0.1, h = 0.01 \; and \; h = 0.001$ respectively.

The results are shown in Tables 1 – 12

IV. DISCUSSION OF RESULTS

Tables 1 – 3 and Tables 7 – 9 are respectively the tables of results and tables of errors for problem 1 (non – stiff IVP) which showed that one - step linear multiderivative method had the best accuracy at l=3, as confirmed by Famurewa and Olorunsola(2013). Furthermore, it can be observed that when h was reduced from 0.1 to 0.01, the accuracy of l=3 improved indicating that better accuracy was achieved at h=0.01 than when h=0.1. Further reduction in the step size to h=0.001 yielded a result with reduced accuracy compared to h=0.01, although, the result at h=0.001 was more accurate than h=0.1.

Tables 4 – 6 and Tables 10 – 12 are respectively the tables of results and tables of errors for problem 2 (stiff IVP) which also showed that one - step linear multiderivative method had the best accuracy at l=3, it was also observed that accuracy was increasing as the step size was reduced from h = 0.1 to h = 0.01 and to h = 0.001.

This study confirmed the accession that reduction in step size improves accuracy of numerical methods but also showed that there is limit to reduction in step size when optimal accuracy is the focus especially for non – stiff initial value problems.

V. CONCLUSION

In this study, it has been established that there is a limit to which the step size *h* of a one - step multiderivative method can be reduced in order to achieve optimal accuracy especially when solving a non – stiff IVP of first order O.D.E.

The study therefore recommended the use of one – step third derivative method (k=1; l=3) with h = 0.01 for solving non – stiff IVPs andh = 0.001 for stiff IVPs of first order O.D.E. in order to achieve more accurate results.

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TABLES OF ERROR FOR PROBLEM 1

TABLE 1: h=0.1

X	L=1	L=2	L=3	L=4	L=5	L=6
0.1	1.5816e-04	3.7540e-08	1.1679e-13	1.4016e-05	4.4140e-03	4.3719e-03
0.2	3.4960e-04	8.2976e-08	2.5823e-13	3.0982e-05	9.7465e-03	9.6538e-03
0.3	5.7958e-04	1.3755e-07	4.2765e-13	5.1360e-05	1.6141e-02	1.59e-02
0.4	8.5408e-04	2.0269e-07	6.3060e-13	7.5682e-05	2.3761e-02	2.3535e-02
0.5	1.1799e-03	2.8001e-07	8.7130e-13	1.0455e-04	3.2793e-02	3.2481e-02
0.6	1.5648e-03	3.7135e-07	1.1555e-12	1.3865e-04	4.3447e-02	4.3035e-02
0.7	2.0177e-03	4.7881e-07	1.4895e-12	1.7878e-04	5.5963e-02	5.5433e-02
0.8	2.5486e-03	6.04e-07	1.8816e-12	2.2580e-04	7.0614e-02	6.9945e-02
0.9	3.1689e-03	7.5192e-07	2.3392e-12	2.8075e-04	8.7709e-02	8.6879e-02
1.0	3.8914e-03	9.2333e-07	2.8728e-12	3.4475e-04	1.0759e-01	1.0657e-01

TABLE 2: h=0.01

X	L=1	L=2	L=3	L=4	L=5	L=6
0.1	1.8144e-06	3.1457e-12	0.0000e+00	1.4662e-07	4.8153e-04	4.8109e-04
0.2	4.0106e-06	6.9531e-12	4.4408e-16	3.2408e-07	1.0642e-03	1.0632e-03
0.3	6.6486e-06	1.1526e-11	4.4408e-16	5.3725e-07	1.7640e-03	1.7624e-03
0.4	9.7972e-06	1.6985e-11	2.2204e-16	7.9167e-07	2.5991e-03	2.5968e-03
0.5	1.3534e-05	2.3465e-11	2.2206e-16	1.0936e-06	3.5902e-03	3.5869e-03
0.6	1.7949e-05	3.1118e-11	0.0000e+00	1.4504e-06	4.7609e-03	4.7565e-03
0.7	2.3143e-05	4.0124e-11	4.4408e-16	1.8701e-06	6.1378e-03	6.1322e-03
0.8	2.9231e-05	5.0678e-11	0.0000e+00	2.3620e-06	7.7516e-03	7.7445e-03
0.9	3.6344e-05	6.3010e-11	0.0000e+00	2.9368e-06	9.6367e-03	9.6279e-03
1.0	4.4629e-05	7.7375e-11	4.4408e-16	3.6062e-06	1.1832e-02	1.1821e-02

TABLE 3: h=0.001

X	L=1	L=2	L=3	L=4	L=5	L=6
0.1	1.8391e-08	2.4424e-15	1.7763e-15	1.4728e-09	4.8579e-05	4.8575e-05
0.2	4.0652e-08	3.9968e-15	5.9952e-15	3.2554e-09	1.0737e-04	1.0736e-04
0.3	6.7391e-08	5.3290e-15	6.4392e-15	5.3967e-09	1.7800e-04	1.7798e-04
0.4	9.9305e-08	5.9952e-15	4.6629e-15	7.9524e-09	2.6229e-04	2.6227e-04
0.5	1.3718e-07	7.5495e-15	7.9936e-15	1.0985e-08	3.6234e-04	3.6231e-04
0.6	1.8193e-07	9.3258e-15	1.0214e-14	1.4569e-08	4.8054e-04	4.8049e-04
0.7	2.3458e-07	1.1990e-14	8.4376e-15	1.8785e-08	6.1958e-04	6.1953e-04
0.8	2.9629e-07	1.3322e-14	1.1102e-14	2.3727e-08	7.8256e-04	7.8249e-04
0.9	3.6838e-07	1.5099e-14	1.3766e-14	2.9500e-08	9.7296e-04	9.7287e-04
1.0	4.5236e-07	1.5987e-14	2.1316e-14	3.6225e-08	1.1947e-03	1.1946e-03

TABLES OF ERROR FOR PROBLEM 2

TABLE 4: h=0.1

X	L=1	L=2	L=3	L=4	L=5	L=6
0.1	1.1687e-01	3.7315e-03	6.4143e-04	3.4237e-03	2.1657e-01	9.9924e-03
0.2	7.1585e-02	4.4407e-03	2.5126e-03	3.3404e-04	1.1681e-01	8.4663e-03
0.3	3.2849e-02	5.8463e-03	5.5211e-03	4.2390e-03	5.7427e-02	6.9310e-03
0.4	1.3081e-02	8.4716e-03	9.60e-03	9.4433e-03	3.8038e-02	7.0740e-03
0.5	4.4293e-03	1.2201e-02	1.4728e-02	1.5337e-02	3.8528e-02	8.6038e-03
0.6	9.0160e-04	1.6879e-02	2.0877e-02	2.2079e-02	4.8461e-02	1.1086e-02
0.7	4.8240e-04	2.2410e-02	2.8044e-02	2.9780e-02	6.3532e-02	1.4244e-02
0.8	1.0131e-03	2.8741e-02	3.6225e-02	3.8500e-02	8.2090e-02	1.7934e-02
0.9	1.2137e-03	3.5848e-02	4.5418e-02	4.8267e-02	1.0351e-01	2.2089e-02
1.0	1.2888e-03	4.3721e-02	5.5625e-02	5.9094e-02	1.2758e-01	2.6677e-02

TABLE 5: h=0.01

X	L=1	L=2	L=3	L=4	L=5	L=6
0.1	3.5160e-04	2.3582e-06	2.9736e-06	2.2258e-05	8.8892e-03	5.4297e-05
0.2	2.5626e-04	7.6472e-06	9.98e-06	7.6291e-06	6.6964e-03	4.7415e-05
0.3	1.3861e-04	1.4361e-05	1.9586e-05	1.15e-05	4.1465e-03	4.2538e-05
0.4	6.4978e-05	2.1957e-05	3.1226e-05	2.9758e-05	2.8620e-03	4.6293e-05
0.5	2.6735e-05	3.0237e-05	4.4713e-05	4.7527e-05	2.6057e-03	5.8005e-05
0.6	8.5676e-06	3.9131e-05	5.9973e-05	6.5847e-05	2.9910e-03	7.5448e-05
0.7	3.7734e-07	4.8612e-05	7.6980e-05	8.5422e-05	3.7680e-03	9.7129e-05
0.8	3.1894e-06	5 -05	9.5724e-05	1.0663e-04	4.8062e-03	1.2226e-04
0.9	4.7051e-06	6.9304e-05	1.1620e-04	1.2965e-04	6.0437e-03	1.5047e-04
1.0	5.3374e-06	8.0512e-05	1.3840e-04	1.5458e-04	7.4529e-03	1.8158e-04

TABLE 6: h=0.001

X	L=1	L=2	L=3	L=4	L=5	L=6
0.1	3.0798e-06	2.0604e-08	2.6136e-08	2.1607e-07	8.2309e-04	5.1823e-07
0.2	2.2457e-06	6.3853e-08	8.5968e-08	8.2464e-08	6.2756e-04	4.6153e-07
0.3	1.2152e-06	1.1578e-07	1.6553e-07	8.8293e-08	3.9354e-04	4.0707e-07
0.4	5.6990e-07	1.7125e-07	2.5969e-07	2.4558e-07	2.7462e-04	4.4268e-07
0.5	2.3456e-07	2.2837e-07	3.6769e-07	3.9333e-07	2.5131e-04	5.5484e-0
0.6	7.5177e-08	2.8644e-07	4.8544e-07	5.4140e-07	2.8856e-04	7.2175e-07
0.7	3.2880e-09	3.4522e-07	6.1607e-07	6.9652e-07	3.63e-04	9.2918e-07
0.8	2.8034e-08	4.0460e-07	7.5837e-07	8.6230e-07	4.6308e-04	1.1696e-06
0.9	4.1351e-08	4.6456e-07	9.1230e-07	1.0405e-06	5.8219e-04	1.4394e-06
1.0	4.6909e-08	5.2508e-07	1.0778e-06	1.2319e-06	7.1790e-04	1.7370e-06