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Applications of Exponentiated Moment Exponential Distribution Based on Dual Generalized Order Statistics

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ABSTRACT

Dual generalized order statistics includes reversed order statistics, lower k-records and lower Pfeifer records. Therefore, the study of characteristics of dual generalized order statistics are of special interest. Pawlas and Szynal (2001) first proposed the concept of dual generalized order statistics (*dgos*) where in the study of distributional properties of random variables by using the inverse image of *gos* is popularly known as dual generalized order statistics. Burkschat *et al.* (2003) further studied this concept in a systematic manner. In this paper, an attempt has been made to derive the exact expression for single as well as for product moments of dual generalized order statistics from exponentiated moment exponential distribution and then results are deduced to reverse order statistics and lower record values. Further, the mean based on dual generalized order statistics and first & second moments based on order statistics from exponentiated moment exponential distribution are computed. Also, we have obtained maximum likelihood estimators of the model parameters based on order statistics. And finally, we have done real data analysis for exhibiting the applications of exponentiated moment exponential distribution in real-world.

Keywords: Dual generalized order statistics, order statistics, exponentiated moment exponential distribution, single & product moments and maximum likelihood estimation.

INTRODUCTION

Over the last three decennium or so, analyst discovered new univariate distributions either due to the practical applications or theoretical considerations or both which are used widely in statistics and associated areas. Present trends and practices in defining probability distributions are different from those proposed before 1997. Now-a-days researchers have shown greater interest in defining new generators or exponentiated classes of univariate continuous distributions by adding one or more parameters or by compounding to build new distributions to a baseline distribution to make the generated distribution more flexible. The idea of exponentiation has been utilized by many authors to propose several exponentiated type of distributions by simply taking the □-th power of the cdf of the classical distributions. For example, Gupta and Kundu (1999) proposed the exponentiated exponential distribution as a generalization of the exponential distribution. Nadarajah and Kotz (2006) discussed the exponentiated type distributions extending the Fréchet, gamma, Gumbel and Weibull distributions. Subsequently, some statistician named it the length biased exponential (LBE) distribution. Later, Hasnain (2013) established the exponentiated moment exponential (EME) distribution as a new generalization of the ME distribution. Khan and Sharma (2016) derived the various properties of Chen distribution based on □□□. Khan and Sharma (2018) discussed the topic of Shannon Entropy and characterization of Nadarajah and Haghighi Distribution based on generalized order statistics. Khan *et.al* (2019) derived some important expressions and huge features of Lindley Distribution based on generalized order statistics.

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A random variable X is said to have three parameters Exponentiated moment exp- distribution (EME), if its probability density function (pdf) is of the form

$$f(x;\beta,\eta,\alpha)=\frac{\alpha(x-\beta)}{\eta^2}e^{-\frac{(x-\beta)}{\eta}}(1-\left(1+\frac{x-\beta}{\eta}\right)e^{-\frac{(x-\beta)}{\eta}})^{\alpha-1},\quad x>\beta. \qquad (1)$$

with the corresponding distribution function (df)

$$F(x,\beta,\eta,\alpha)=(1-\left(1+\frac{x-\beta}{\eta}\right)e^{-\frac{(x-\beta)}{\eta}})^{\alpha} \qquad (2)$$

where, $\beta\ \in\Re$, η and $\alpha>0$ are the location, scale and shape parameters respectively. Introducing new parameters such as location, scale, shape and inequality in the existing distributions in order to add more flexibility has been adopted by several authors in the last 30 years or so. Putting $\alpha=1$, the EME distribution reduces to the ME distribution. Setting β=0 and η=1 in (1) and (2) the pdf and cdf of the standard EME distribution are obtained as

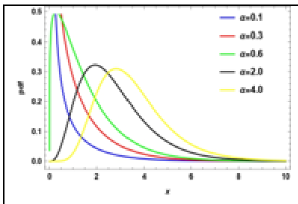
$$f(x)=\alpha x e^{-x}(1-(1+x)e^{-x})^{\alpha-1},\ \ x>0 \qquad (3)$$

$$F(x)=[1-(1+x)e^{-x}]^{\alpha}. \qquad (4)$$

and reversed hazard rate function is given here

$$h(x)=\frac{\alpha x e^{-x}}{[1-(1+x)e^{-x}]} \qquad (5)$$

The reverse hazard rate function is the ratio of the probability density to its distribution function and is defined by $h(x)=f(t)/F(x)$, and has applications in the field of engineering, actuarial science, forensic science. Graphical representation of the pdf (3), distribution function (4) for the standard EME distribution and the hazard rate function (5) respectively for different parameters value is given below. These plots show that the pdf can be right skewed, approximately symmetric or reversed J-shape and we can see that the reversed hrf can be increasing and decreasing-increasing (bathtub-shaped) depending on the value of alpha, which makes it very flexible for modelling lifetime phenomena Henceforth, X denotes a random variable with pdf (1).



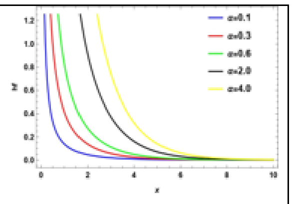


Figure 1: pdf and hf of the various parameter values

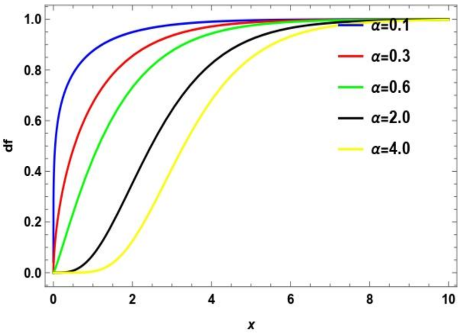


Figure 1: df of the above-mentioned distribution

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Model Description

The random variables $X(1,n,m,k), X(2,n,m,k), \dots, X(n,n,m,k)$ are said to be dual generalized order statistics ($dgos$) from a continuous population with the $df\ F(x)$ and the $pdf\ f(x)$ if their joint pdf has the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f(x_i)\right)[F(x_n)]^{k-1} f(x_n)$$

for $x_1 > x_2 > \cdots > x_n$. where $n \geq 2$, $k > 0$ and

$-\infty \leq$ m< $\leq \infty$ are such that $\gamma_r = k + (n-r)(m+1) > 0$ for all $r = 1,2,\dots,n-1$. It follows that the pdf of $r-$th $dgos\ X(r,n,m,k)$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)) \tag{6}$$

and the joint pdf of $r-$th and $s-$th, $dgos\ 1 \leq r < s \leq n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\ (s-r-1)!}[F(x)]^m g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(x) f(y), \ \ y < x, \tag{7}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \gamma_i = k + n - i + \sum_{j=1}^{n-i} m_j = k + (n-i)(m+1)$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \quad m \neq -1 \\ -\ln(1-x) & , \quad m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \ \ x \in [0,1).$$

The model of $dgos$ contains special cases such as ordinary reverse order statistics ($\gamma_i = n - i + 1; i = 1,2,\dots,n\ i.e.\ m_1 = m_2 = \cdots = m_{n-1} = 0, k = 1$), k-th lower record values ($\gamma_i = k\ i.e.\ m_1 = m_2 = \cdots = m_{n-1} = -1, k \in N$), sequential order statistics ($\gamma_i = (n-i+1)\alpha_i; \alpha_1, \alpha_2, \dots, \alpha_n > 0$) and progressive type II censored order statistics ($m_i \in N, k \in N$) [Kamps (1995)]. For more details about application of dgos and its sub models, one may see Khan *et al.* (2019a, 2019b), Singh *et al.* (2022, 2023a, 2023b), Ahsanullah (2004, 2005), Akhter *et al.* (2022). Further, we shall use the following integral formula of Gradshteyn and Ryzhik (2000) to prove some results of this article.

$$\int_0^\infty x^{\nu-1} e^{-\eta x} = \frac{\Gamma\nu}{\eta^\nu} \tag{8}$$

$$\int_\mu^\infty x^{\nu-1} e^{-\eta x} dx = \eta^{-\nu}\Gamma(\nu, \eta u); [\mu > 0, \eta > 0] \tag{9}$$

$$\int_0^\infty x^{\nu-1} e^{-\eta x}\Gamma(\tau, \omega x) dx = \frac{\omega^\tau \Gamma(\nu+\tau)}{\nu(\omega+\eta)^{\nu+\tau}} 2F_1(1, \nu+\tau; \nu+1; \frac{\eta}{\omega+\eta}) \tag{10}$$

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where $[Re(\omega+\eta)>0, Rev>0, Rev+\tau>0]$, and $2F_1(a,b;c;x)$ denotes the Gauss Hypergeometric function defined by $2F_1(a,b;c;x)=\sum_{p=0}^{\infty}\frac{(a)_p(b)_p x^p}{(c)_p p!}$ with $(e_p)=e(e+1)\ldots(e+p+1)$; is the ascending factorial.

Single Moment of Eme Distribution Based on *Dgos*

In this section, we have derived the explicit expression for Single Moment of EME distribution based on $dgos$.

Theorem 2.1 Let $X_1, X_2, \ldots, X_n$ be n continuous iid non negative random variables and follow exponentiated moment exponential distribution as given in (1) and $n \in N, \widetilde{m} \in \mathbb{R}, k>0, 1 \le r < s \le n, i,j=1,2,\ldots$,then the single moment of r-th *dgos* is given by

$$E(X^j(r,n,m,k))=\frac{\alpha\eta^a\beta^{j-a}C_{r-1}}{(r-1)!\,(m+1)^{r-1}}\sum_{u=0}^{r-1}\sum_{a=0}^{j}\sum_{p=0}^{\alpha\gamma_{r-u}-1}\sum_{q=0}^{p}\binom{j}{a}\binom{r-1}{u}$$
$$\times\binom{\alpha\gamma_{r-u}-1}{p}(-1)^{u+p}\binom{p}{q}\frac{\Gamma(a+q+2)}{(1+p)^{a+q+2}}. \qquad (11)$$

Proof: From (6), we get

$$E[X(r,n,m,k)]^j=\frac{C_{r-1}}{(r-1)!}\int_{\beta}^{\infty}x^j\,[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx.$$

On application of (1) and (2) equivalently in the above equation, we get

$$E[X(r,n,m,k)]^j=\frac{\alpha C_{r-1}}{\eta(r-1)!\,(m+1)^{r-1}}\sum_{u=0}^{r-1}\binom{r-1}{u}(-1)^u\int_{\beta}^{\infty}x^j\left(\frac{x-\beta}{\eta}\right)e^{-\left(\frac{x-\beta}{\eta}\right)}$$
$$\times[1-\left(1+\left(\frac{x-\beta}{\eta}\right)\right)e^{-\left(\frac{x-\beta}{\eta}\right)}]^{\alpha\gamma_{r-u}-1}dx. \qquad (12)$$

Let $\left(\frac{x-\beta}{\eta}\right)=t$ in (12), we have

$$=A\alpha\int_0^{\infty}(\beta+\eta t)^j t e^{-t}[1-(1+t)e^{-t}]^{\alpha\gamma_{r-u}-1}dt \qquad (13)$$

where,

$$A=\frac{C_{r-1}}{(r-1)!\,(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}$$

Now using binomial series in (13), simplifying it and hence we have the given expression

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$$E(X^j(r,n,m,k)) = A\alpha\eta^a \sum_{a=0}^{j} \sum_{p=0}^{\alpha\gamma_{r-u}-1} \sum_{q=0}^{p} \binom{j}{a} \binom{\alpha\gamma_{r-u}-1}{p} \times \binom{p}{q} (-1)^p \int_0^\infty t^{a+q+1} e^{-(p+1)t} dt \qquad (14)$$

Using gamma function (14), we have the required result given in the (11).

When $m = -1$

$$E[X(r,n,-1,k)]^j = \frac{k^r}{(r-1)!} \int_\beta^\infty x^j \, [-lnF(x)]^{r-1} [F(x)]^{k-1} f(x) dx.$$

On using (1) and (2) in the above equation

$$E[X(r,n,-1,k)]^j = \frac{(\alpha k)^r}{\eta^2 (r-1)!} \int_\beta^\infty x^j \, (x-\beta)[-ln(1-\left(1+\frac{x-\beta}{\eta}\right) e^{-}\frac{(x-\beta)}{\eta})^\alpha]^{r-1}$$

$$\times e^{-}\frac{(x-\beta)}{\eta} (1-\left(1+\frac{x-\beta}{\eta}\right) e^{-}\frac{(x-\beta)}{\eta})^{\alpha k-1} dx \qquad (15)$$

By setting $\left(\frac{x-\beta}{\eta}\right) = t$ in (15), we find that

$$= \frac{(k\alpha)^r}{(r-1)!} \int_0^\infty (\beta+\eta t)^j t e^{-t} (-ln[1-(1+t)e^{-t}])^{r-1} [1-(1+t)e^{-t}]^{\alpha k-1} dt.$$

Using the logarithmic expansion

$$[-ln(1-t)]^j = (\sum_{w=1}^{\infty} \frac{t^w}{w})^j = \sum_{w=0}^{\infty} Z_w \, (j) t^{j+w}, |t| < 1,$$

where $Z_w(j)$ is the coefficient of t^{j+w} in the expansion of $(\sum_{w=1}^{\infty} \frac{t^w}{w})^j$ [see Balakrishnan and Cohen (1991), Shawky and Bakoban (2008)] and simplifying the resulting expression, we get

$$E(X^j(r,n,-1,k)) = \frac{(k\alpha)^r \eta^a \beta^{j-a}}{(r-1)!} \sum_{a=0}^{j} \sum_{b=0}^{\alpha k-1} \sum_{w=0}^{\infty} \sum_{c=0}^{b+r+w-1} \binom{j}{a} \binom{\alpha k-1}{b} \times \binom{b+r+w-1}{c} (-1)^b Z_w(r-1) \int_0^\infty t^{a+c+1} e^{-(b+r+w)t} dt \qquad (16)$$

Now applying the gamma function (16) in the above equation, we get the result for the k-th lower record values from EME distribution given below:

$$E(X^j(r,n,-1,k)) = \frac{(k\alpha)^r \eta^a \beta^{j-a}}{(r-1)!} \sum_{a=0}^{j} \sum_{b=0}^{\alpha k-1} \sum_{w=0}^{\infty} \sum_{c=0}^{b+r+w-1} \binom{j}{a} \binom{\alpha k-1}{b}$$

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$$\times \binom{b+r+w-1}{c} (-1)^b Z_w(r-1) \frac{\Gamma(c+a+2)}{(b+r+w)^{c+a+2}} \tag{17}$$

Remark 1. Setting $m=0$, $k=1$ in (17), the exact explicit formula for single moments of order statistics can be obtained as

$$E(X_{n-r+1:n})^j = \frac{\alpha\eta^a\beta^{j-a}n!}{(r-1)!\,(n-r)!}\sum_{u=0}^{r-1}\sum_{a=0}^{j}\sum_{p=0}^{\alpha(n-r+u+1)-1}\sum_{q=0}^{p}\binom{j}{a}\binom{r-1}{u}$$

$$\times \binom{\alpha(n-r+u+1)-1}{p}(-1)^{u+p}\binom{p}{q}\frac{\Gamma(a+q+2)}{(1+p)^{a+q+2}}.$$

$$E(X_{r:n})^j = \frac{\alpha\eta^a\beta^{j-a}n!}{(r-1)!\,(n-r)!}\sum_{u=0}^{n-r}\sum_{a=0}^{j}\sum_{p=0}^{\alpha(r+u)-1}\sum_{q=0}^{p}\binom{j}{a}\binom{n-r}{u}$$

$$\times \binom{\alpha(r+u)-1}{p}(-1)^{u+p}\binom{p}{q}\frac{\Gamma(a+q+2)}{(1+p)^{a+q+2}}. \tag{18}$$

Remark 2. Let k = 1 in (18), we deduce the exact expression for the single moment of lower record for EME distribution

$$E(X^j(r,n,-1,1)) = \frac{\alpha^r\eta^a\beta^{j-a}}{(r-1)!}\sum_{a=0}^{j}\sum_{b=0}^{\alpha-1}\sum_{w=0}^{\infty}\sum_{c=0}^{b+r+w-1}\binom{j}{a}\binom{\alpha-1}{b}\binom{b+r+w-1}{c}$$

$$\times (-1)^b Z_w(r-1)\frac{\Gamma(c+a+2)}{(b+r+w)^{c+a+2}}$$

Special Case 1. On putting $\beta=0$ and $\eta=1$ in (18), we have the exact expression for single moments of order statistics from the standard EME distribution as obtained by Akhter *et al.*(2022).

Product Moment of Eme Distribution Based on *Dgos*

In this section, we have found the explicit expression for product moment of EME distribution based on $dgos$.

Theorem 2.1 Let $X_1, X_2, \ldots, X_n$ be n continuous iid non-negative random variables and follow exponentiated moment exponential distribution as given in (1) and $n \in N, \tilde{m} \in \mathbb{R}, k>0, 1 \le r < s \le n, i,j = 1,2,\ldots$,then the product moment based on $dgos$ is given by

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j] = \frac{\alpha^2\eta^{a+h}\beta^{i+j-a-h}C_{s-1}}{(r-1)!\,(s-r-1)!\,(n-s)!\,(m+1)^{s-2}}$$

$$\times \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{h=0}^{j}\sum_{d=0}^{b}\sum_{q=0}^{p}\sum_{a=0}^{i}\sum_{p=0}^{\alpha(s+u-r-v)(m+1)-1}\sum_{b=0}^{\alpha\gamma_{s-v}-1}(-1)^{u+v+p+b}\binom{r-1}{u}$$

$$\times \binom{s-r-1}{v}\binom{\alpha(s+u-r-v)(m+1)-1}{p}\binom{p}{q}\binom{b}{d}\binom{i}{a}\binom{j}{h}$$

$$\times (-1)^{u+v+p+b} \frac{\Gamma h+d+a+q+4}{(h+d+2)(p+b+2)^{h+d+a+q+4}}$$

$$\times\, 2F_1\left(1, h+d+a+q+4; h+d+2+1; \frac{b+1}{p+b+2}\right)$$

Proof: The joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, (7) can be written as

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j] = \frac{C_{s-1}}{(r-1)!\ (s-r-1)!\,(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}$$

$$\times \binom{r-1}{u}\binom{s-r-1}{v} \int_\beta^\infty \int_y^\infty x^i\, y^j [F(x)]^{(s+u-r-v)(m+1)-1}$$

$$\times [F(y)]^{\gamma_{s-v}-1} f(x)\, f(y) dxdy, \qquad y < x,$$

$$= A\int_\beta^\infty y^j\, [F(y)]^{\gamma_{s-v}-1} f(y) I(y) dy \tag{19}$$

where

$$A = \frac{C_{s-1}}{(r-1)!\ (s-r-1)!\,(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} \binom{r-1}{u}\binom{s-r-1}{v}$$

and

$$I(y) = \int_y^\infty x^i\, [F(x)]^{(s+u-r-v)(m+1)-1} f(x) dx$$

Now put the $F(x)$ from (2) in above equation, we have

$$I(y) = \sum_{p=0}^{\alpha((s+u-r-v)(m+1))-1} \frac{(-1)^p\alpha}{\eta} \binom{\alpha(s+u-r-v)(m+1)-1}{p}$$

$$\times \int_y^\infty x^i\, (1+\frac{x-\beta}{\eta})^p \left(\frac{x-\beta}{\eta}\right) e^{-(p+1)\left(\frac{x-\beta}{\eta}\right)} dx \tag{20}$$

Let $\left(\frac{x-\beta}{\eta}\right) = t$ in (20), we get

$$I(y) = \sum_{a=0}^{i}\sum_{q=0}^{p}\sum_{p=0}^{\alpha(s+u-r-v)(m+1)-1} (-1)^p \alpha\beta^{i-a}\eta^a \binom{\alpha(s+u-r-v)(m+1)-1}{p}$$

$$\times \binom{i}{a}\binom{p}{q} \int_{\frac{y-\beta}{\eta}}^\infty t^{a+q+1}\, e^{-(p+1)t} dt \tag{21}$$

On application of (9) in (21), we have below simplified expression

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$$I(y) = \sum_{a=0}^{i}\sum_{q=0}^{p}\sum_{p=0}^{\alpha(s+u-r-v)(m+1)-1} (-1)^p \beta^{i-a}\eta^a \alpha \binom{i}{a}\binom{\alpha(s+u-r-v)(m+1)-1}{p}$$

$$\times \binom{p}{q}(p+1)^{-(a+q+2)}\Gamma a+q+2,(p+1)\left(\frac{y-\beta}{\eta}\right)$$

Now we put the value of I(y) in (19), obtained result given below

$$= A\sum_{a=0}^{i}\sum_{q=0}^{p}\sum_{p=0}^{\alpha(s+u-r-v)(m+1)-1} (-1)^p \beta^{i-a}\eta^a \alpha \binom{\alpha(s+u-r-v)(m+1)-1}{p}\binom{p}{q}$$

$$\times \binom{i}{a}(p+1)^{-(a+q+2)}\int_{\beta}^{\infty} y^j\,[F(y)]^{\gamma_{s-v}-1}, f(y)\Gamma a+q+2,(p+1)\left(\frac{y-\beta}{\eta}\right)dy \qquad (22)$$

On application of (1) and (2) in (22) and then assume $\left(\frac{y-\beta}{\eta}\right) = t$ in simplified form, we have

$$= A\alpha \sum_{p=0}^{\alpha((s+u-r-v)(m+1))-1}\sum_{q=0}^{p}\binom{p}{q}(-1)^p\binom{\alpha(s+u-r-v)(m+1)-1}{p}$$

$$\times (p+1)^{-(a+q+2)}\sum_{a=0}^{i}\beta^{i-a}\,\eta^a\alpha$$

$$\times \int_0^{\infty}(\beta+t\eta)^j te^{-t}[1-(1+t)e^{-t}]^{\alpha\gamma_{s-v}-1}\Gamma(a+q+2),(p+1)tdt \qquad (23)$$

Now expand the above expression binomially and hence we have the required result given here,

$$= A\sum_{p=0}^{\alpha((s+u-r-v)(m+1))-1}\sum_{q=0}^{p}(-1)^p\alpha\binom{\alpha(s+u-r-v)(m+1)-1}{p}\binom{p}{q}$$

$$\times (p+1)^{-(a+q+2)}\sum_{a=0}^{i}\beta^{i-a}\,\eta^a\alpha\int_0^{\infty}(\beta+t\eta)^j te^{-t}$$

$$\times [1-(1+t)e^{-t}]^{\alpha\gamma_{s-v}-1}\Gamma(a+q+2),(p+1)tdt$$

$$= A\sum_{p=0}^{\alpha((s+u-r-v)(m+1))-1}\sum_{q=0}^{p}(-1)^p\alpha\binom{\alpha(s+u-r-v)(m+1)-1}{p}$$

$$\times \binom{p}{q}(p+1)^{-(a+q+2)}\sum_{h=0}^{j}\sum_{b=0}^{\alpha\gamma_{s-v}-1}\sum_{d=0}^{b}\binom{j}{h}\beta^{j+i-a-h}\alpha\eta^{a+h}\binom{\alpha\gamma_{s-v}-1}{b}$$

$$\times \binom{b}{d}(-1)^b\int_0^{\infty}t^{h+d+1}\,e^{-(b+1)t}\Gamma(a+q+2),(p+1)tdt. \qquad (24)$$

Now using (10) in (23), putting the value of A, we have the required result given in (24)

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Remark 4.1. Setting $m = 0$, $k = 1$ in (24), the exact explicit formula for single moments of order statistics can be obtained as

$$E[(X_{(n-r+1:n)})^i (X_{(n-s+1:n)})^j] = \alpha^2 \eta^{a+h} \beta^{i+j-a-h} C_{r,s:n} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} \sum_{a=0}^{i} \sum_{h=0}^{j} \sum_{d=0}^{b} \sum_{q=0}^{p}$$

$$\times \sum_{p=0}^{\alpha(s+u-r-v)-1} \sum_{b=0}^{\alpha(n-s+v+1)-1} (-1)^{u+v+p+b} \binom{r-1}{u} \binom{s-r-1}{v} \binom{\alpha(s+u-r-v)-1}{p}$$

$$\times \binom{j}{h} \frac{\Gamma h+d+a+q+4}{(h+d+2)(p+b+2)^{h+d+a+q+4}} 2F_1\left(1, h+d+a+q+4; h+d+2+1; \frac{b+1}{p+b+2}\right)$$

Numerical Computation for Moments

In this section, we have derived the exact and explicit expressions for single and product moment of exponentiated moment exponential distribution based on dual generalized order statistics. Further, by putting the specific value of the parameters □ and □, we have deduced single moment, product moment of order statistics and lower records from Exponentiated moment exponential distribution. The below tables have presented the expected values, first and second moments based on order statistics and as well as on dual generalized orderv statistics from exponentiated moment exponential distribution. For n=1,2,...,10 and □=1.0,1.5,2.0,2.5,3.0,3.5,4.0 and 4.5. One can see in Table 1, the mean decreasing both with respect to n and r. In Table 2 and Table 3 we have reported the first and second moments based on order statistics, from these tables we can calculate variances based on the order statistics. All computations are performed using Mathematica and Matlab. Mathematica allow us to do algebraic manipulation packages for better arbitrary precisions, so the accuracy of the given values is not an issue.

Table 1: Mean of dual generalized order statistics

Table 2: Mean of dual generalized order statistics

n	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$
1	2.00									
2	1.07099	1.39352								
3	0.761654	0.773333	1.1619							
4	0.600505	0.564023	0.648304	1.0314						
5	0.4999	0.452721	0.47679	0.575131	0.944647					
6	0.430448	0.381985	0.385518	0.424343	0.52559	0.881434				
7	0.379307	0.33242	0.327294	0.344348	0.388253	0.489141	0.832602			

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8	0.339911	0.295466	0.286313	0.293328	0.315658	0.361419	0.460834	0.793323		
9	0.308535	0.266696	0.255619	0.25738	0.269425	0.294136	0.340432	0.438002	0.7607	
11	0.282896	0.243574	0.231619	0.230411	0.236859	0.251363	0.277193	0.323418	0.4190	0.733203

Table 3: First moments of order statistics

n	r	$\theta = 1.0$	$\theta = 1.5$	$\theta = 2.0$	$\theta = 2.5$	$\theta = 3.0$	$\theta = 3.5$	$\theta = 4.0$	$\theta = 4.5$
1	1	2.0000	3.0000	2.7500	2.6562	3.21296	3.6377	3.5470	3.49541
2	1	1.25	2.78704	1.95284	1.50423	2.40361	3.07193	2.73374	2.49172
	2	2.75	3.21296	3.54716	3.80827	4.02231	4.20353	4.36059	4.4991
3	1	0.96	2.85652	1.63082	0.939218	2.07106	2.90535	2.39682	2.02365
	2	1.82407	2.64807	2.59687	2.63425	3.06873	3.40509	3.40759	3.42789
	3	3.21	3.49541	4.02231	4.39528	4.4991	4.60276	4.83709	5.03473
4	1	0.804687	2.68155	1.44567	0.733548	1.8773	2.72289	2.19928	1.81404
	2	1.43778	3.38144	2.18625	1.55623	2.65232	3.45273	2.98944	2.65247
	3	2.21035	1.9147	3.00749	3.71227	3.48515	3.357457	3.82573	4.20323
	4	3.547164	4.02231	4.36059	4.62296	4.83709	5.01786	5.1742	5.31189
5	1	0.70208	2.10819	1.3215	0.857061	1.74595	2.41586	2.06468	1.81115
	2	1.215117	4.97499	1.94237	0.239493	2.4027	3.95101	2.7377	1.82559
	3	1.77179	0.991121	2.55207	3.53133	3.02674	2.7053	3.36704	3.8928
	4	2.50274	2.53042	3.3111	3.83289	3.79075	3.79222	4.13153	4.41018

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	5	3.80828	4.39528	4.62296	4.82047	5.09867	5.32427	5.43487	5.53732

We can note that the fact $\sum_{i=0}^{n} E\,(X_{i:n}^{j}) = nE(X)^{j}$ (David and Nagaraja (2003)) is satisfied here.

Table 4: Second moments of order statistics

n	r	$\theta = 1.0$	$\theta = 1.5$	$\theta = 2.0$	$\theta = 2.5$	$\theta = 3.0$	$\theta = 3.5$	$\theta = 4.0$	$\theta = 4.5$
1	1	6.0000	9.0000	9.7500	10.7813	12.5586	14.0189	14.8322	15.6486
2	1	2.2500	5.44136	4.6677	4.80697	6.68765	8.12145	8.40794	8.81867
	2	9.75	12.5586	14.8322	16.7555	18.4296	19.9164	21.2565	22.4785
3	1	1.30864	4.97268	3.18296	2.75929	4.86555	6.39559	6.34588	6.50794
	2	4.13272	6.37872	7.6374	8.90231	10.3319	11.5732	12.5322	13.4401
	3	12.5586	15.6486	18.4296	20.6821	22.4785	24.088	25.6187	26.9977
4	1	0.902344	4.81292	2.4687	1.71725	3.95218	5.53843	5.28882	5.30233
	2	2.52754	5.45195	5.32577	5.88542	7.06563	8.96704	9.51688	10.1248
	3	5.7379	7.3055	9.94903	11.9192	13.0581	14.1793	15.5474	16.7555
	4	14.8322	18.4296	21.2565	23.6031	25.6187	27.3908	28.9758	30.4118
5	1	0.680832	4.43355	2.04471	1.20408	3.39271	4.94148	4.63039	4.59166
	2	1.78839	6.33037	4.16463	3.76996	6.19005	7.92624	7.92253	8.14502
	3	3.63626	4.13432	7.06749	9.0586	9.7290	10.5282	11.9084	13.0944
	4	7.13899	9.41961	11.8701	13.8263	15.2774	16.6134	17.9734	19.1962
	5	16.7555	20.6821	23.6031	26.0473	28.204	30.0852	31.7264	33.2157

Variance of order statistics can by computed by using the following formula:

$$(\sigma_{r:n})^2 = (\mu_{r:n})^{(2)} - (\mu_{r:n})^{(1)}$$

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Maximum Likelihood Estimation Based on *Dgos*

In this section, we have exhibited the expression for MLE of the shape parameters of the Exponentiated moment exponential distribution based on $dgos$. Estimation of parameters through MLE based on gos and dgos has always been the topic of interest among researchers. In view of (1) and (2), the likelihood function for Exponentiated moment exponential distribution based on $dgos$ is given by equation (25) as follows:

$$L(\alpha,\beta) = k\alpha\beta^2 \prod_{j=1}^{n-1}\gamma_j \prod_{i=1}^{n}\left(x_i e^{-\frac{x_i}{\beta}}\right)\prod_{i=1}^{n-1}[1-\left(1+\frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}]^{\alpha(m_i+1)-1}$$

$$\times [1-\left(1+\frac{x_n}{\beta}\right)e^{-\frac{x_n}{\beta}}]^{\alpha k-1} \tag{25}$$

After taking log on both sides, we get

$$lnL(\alpha,\beta) = \ln k + n[\ln\alpha - 2\ln\beta] + \sum_{j=1}^{n-1} l\, n\gamma_j + \sum_{i=1}^{n}\left(\ln x_i - \frac{x_i}{\beta}\right)$$

$$-\sum_{i=1}^{n-1}[\,\alpha(m_i+1)-1]\ln\left(1-\left(1+\frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}\right)$$

$$+(\alpha k-1)\ln\left(1-\left(1+\frac{x_n}{\beta}\right)e^{-\frac{x_n}{\beta}}\right)$$

The exact expression for MLE of α and β cannot be obtained directly. Therefore, an iteration method (Newton type algorithm) is applied to obtain the MLE of α and β.

Simulation Study

In this section, we have estimated the parameters (α,β) of exponentiated moment exponential distribution by the method of maximum likelihood estimation. We have also computed the bias and Mean Square Error (MSEs) for enhancing the performance of ML estimator. All simulations are performed by R software with nlm function. In entire simulation process, we consider 10000 replications of the process and the average estimate, bias, and mean square error (MSE) are calculated based on order statistics. In each replication, a random sample of size $n = 20,40,60,80,$ and 100 is drawn with different combinations of (α,β)=(2,2.2), (2.5,2.7), (3,3.2), (3.5,3.7), (4,4.2) from the proposed distribution. In Table 4, we have shown the MLEs of parameters (α,β), their biases (Bias) and mean square errors (MSE) from exponentiated moment exponential distribution.

Table 5: MLE, Bias and MSEs of exponentiated moment exponential distribution based on order statistics

α,β	N	MLE	Bias	MSE	MLE	Bias	MSE
(2,2.2)	**20**	2.258316	0.2583162	0.06672726	2.316214	0.1162135	0.01350558
	40	2.242036	0.24203553	0.058581197	2.288927	0.08892656	0.007907934
	60	2.149757	0.14975741	0.022427281	2.267224	0.06722372	0.004519028

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	80	2.116832	0.11683195	0.00676989	2.225082	0.02508175	0.004703453
	100	2.006837	0.006837	0.000046	2.219457	0.01945736	0.000378589
(2.5,2.7)	20	2.670285	0.1702852	0.02899704	2.953296	0.2532955	0.06415862
	40	2.604219	0.1042188	0.01086155	2.840427	0.1404272	0.0197198
	60	2.590082	0.09008207	0.008114779	2.821723	0.12172303	0.014816497
	80	2.522039	0.02203858	0.000485699	2.741043	0.04104304	0.001684531
	100	2.502297	0.002297	0.000005275	2.711197	0.01119717	0.000125377
(3,3.2)	**20**	3.150254	0.1502537	0.02257618	3.497605	0.297605	0.08856875
	40	3.107674	0.1076736	0.0115936	3.41182	0.2118198	0.04486763
	60	3.085461	0.08546	0.007303412	3.360349	0.1603492	0.025711866
	80	3.062624	0.06262438	0.003921813	3.243575	0.04357537	0.001898813
	100	3.021351	0.02135136	0.00045588	3.212411	0.01241144	0.000154044
(3.5,3.7)	**20**	3.906878	0.4068781	0.16554978	3.921959	0.2219589	0.04926576
	40	3.791089	0.2910893	0.08473297	3.914458	0.2144584	0.04599242
	60	3.719279	0.2192791	0.04808333	3.882171	0.1821711	0.03318631
	80	3.629095	0.12909534	0.016665608	3.754097	0.05409673	0.002926456
	100	3.568181	0.06818069	0.004648606	3.75239	0.05239031	0.002744744
(4,4.2)	**20**	4.21549	0.2154904	0.04643612	4.449151	0.245191	0.06011863
	40	4.139653	0.1396533	0.01950305	4.362092	0.162092	0.02627382
	60	4.125825	0.125825	0.01583193	4.239339	0.03933917	0.00154757

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	80	4.080527	0.08052652	0.006484521	4.222211	0.022211	0.000493329
	100	4.068569	0.06856867	0.004701663	4.215117	0.015117	0.000228524

Real Data Analysis

In this section, we have considered two data. First one is Air quality data of Delhi which shows the Air quality index of Delhi in 12 months duration which are observed in 2021 and second is the State-wise GDP of India (2025) provides an extensive breakdown of the GDP of each state. The fitting of the proposed model based on dual generalized order statistics has been shown with the following distributions:

- Generalized Exponential (GE)
- Exponentiated Exponential (EE)
- Inverted Kumaraswamy (IK)

The fitting compatibility has been performed by comparing the goodness-of-fit of the models based on dual generalized order statistics criteria with negative likelihood, Akaike information criterion (AIC), corrected AIC, Hannan–Quinn information criterion (HQIC), Bayesian Information Criterion (BIC). The model with least AIC, CAIC, HQIC and BIC is treated as best model. Further, we get the Kolmogorov-Smirnov (K-S) statistic. In general, the smaller the value of K-S statistic, the better the fit to the data. The comparison of these features is exhibited in Table 5 by using data 1 and in Table 7 by using data 2 with Exponentiated moment exponential distribution. This comparison absolutely proved that EME distribution is best choice among various continuous distribution. Further, the data fitted density are shown in the figure 3 and figure 4 for data 1 and data 2, respectively with histogram of the data and empirical cumulative distribution plots which exhibits that the proposed model of EME distribution provides adequate fitting to the considered data. All aspects of real data analysis are computed by using R software.

analysis with data 1

This dataset falls under the category Environmental Data-Air Quality Data. It sourced from Kaggle dataset contains estimated pollutant concentrations (PM2.5, PM10, NO2, SO2, CO, and O3). We have used CO (mg/m^3) quantity data for our data analysis. The data values are as follows: 1.3770161, 1.1838053, 0.8772581, 1.0535833, 0.9490323, 0.7578333, 0.6651613, 0.5557258, 0.6335833, 0.9845968, 1.8650833, 1.4279839.

Table 6: Comparison of model features with other distributions

Model	MLE	-loglikelihood	AIC	CAIC	HQIC	BIC	KS
EME	(3.623 0.241)	11.59765	27.19529	28.52863	26.83623	28.1651	0.29035
GE	(9.191 3.432)	12.25593	28.51187	29.8452	28.15281	29.48168	0.29548
EE	(2.0 2.5)	35.00596	74.01193	75.34526	73.65287	74.98174	0.56363

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IK	(2 0.101)	43.36172	90.72344	92.05678	90.36438	91.69326	0.94748

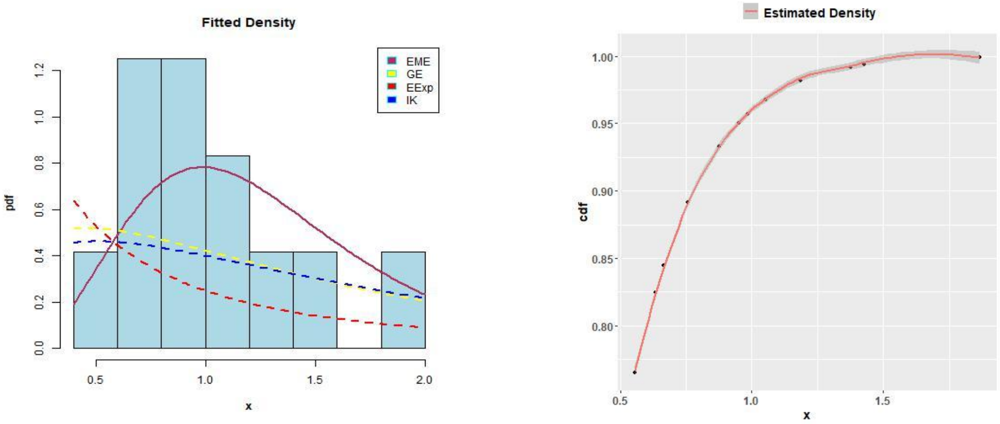

Figure 2: The fitted density (left) and estimated *cdf* (right) for CO (mg/m^3) quantity of Delhi's Air quality data

Analysis with DATA 2

Gross Domestic Product or GDP is a measure of the total value of all goods and services produced within a country or region in a specific time frame. It is a vital indicator of economic health. A high GDP indicates strong production, employment and income levels, making it crucial for evaluating an economic development of a state or a nation. So we have taken second data which contains GDP of Indian states in ₹ Crore (2024-25) as follows:

Table 7: GDP of Indian states in ₹ Crore (2024-25)

State	GDP Value	State	GDP Value
Maharashtra	24.11 Lakhs	Kerala	6.35 Lakhs
Tamil Nadu	15.71 Lakhs	Haryana	6.34 Lakhs
Uttar Pradesh	14.23 Lakhs	Punjab	4.96 Lakhs
Karnataka	14.23 Lakhs	Odisha	5.21 Lakhs
West Bengal	9.04 Lakhs	Bihar	4.65 Lakhs
Rajasthan	8.45 Lakhs	Assam	3.19 Lakhs
Telangana	7.93 Lakhs	Jharkhand	2.85 Lakhs
Andhra Pradesh	8.21 Lakhs	Chhattisgarh	3.22 Lakhs
Madhya Pradesh	6.6 Lakhs	Uttarakhand	2.13 Lakhs
Delhi	6.72 Lakhs	-	-

Table 8: Comparison of model features with other distributions

Model	MLE	-loglikelihood	AIC	CAIC	HQIC	BIC	KS
EME	(0.775 3.275)	66.86649	137.733	138.483	138.0527	139.6219	0.29532
GE	(1.749 0.245)	67.39207	138.7841	139.5341	139.1038	140.673	0.29778
Ik	(1.99 0.101)	147.2949	298.5899	299.3399	298.9096	300.4788	0.98916
EE	(2.0 2.5)	385.3441	774.6882	775.4382	775.0079	776.5771	0.99029

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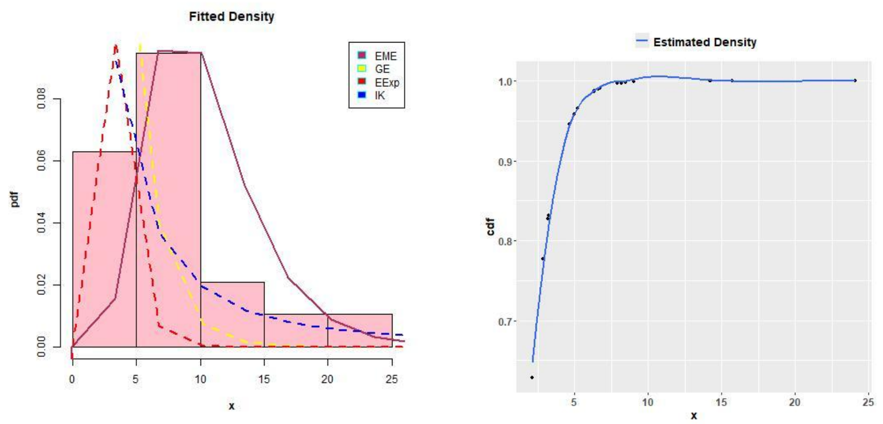

Figure 3: The fitted density (left) and estimated *cdf* (right) for GDP of Indian states data

CONCLUSION

In this paper, we have derived exact and explicit expressions for single and product moments based on dual generalized order statistics for exponentiated moment exponential distribution. By evaluating the parameters of *dgos*, we have found the exact and explicit expressions of order statistics and lower record values for single moments. But in case of product moment, we have executed only exact moments based on order statistics; the lower record values could not be derived. By utilizing these expressions, we have calculated the single moments of Exponentiated moment exponential distribution based on order statistics and *dgos* for different values of distribution's parameters. Further highlighting the distribution properties, the simulation study is also discussed with the help of MLE method and R software based on order statistics. And it is concluded that the average bias and MSEs for individual parameters of the given distribution fall close to zero when sample size increases, which enlightens the consistency of distribution's estimators.

At last, we have included real-world applications for examining that the EMED is the best in terms of fit and practicality in comparison of various continuous distribution.

Overall, our findings enlighten the excellent features of the EMED models by analyzing some data sets and we can apply these applications in various fields.

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Disclosure Statement

There are no conflicts of interest to disclose.

Data Availability Statement

We have analyzed two data in this article in which first data is available at https://github.com/zeeshanDA/Delhi-Air-Quality-Dashboard-2021-2024- and second data is available at https://pwonlyias.com/stage/gdp-of-indian-states.

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