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Evaluation of Some Special Integrals by Decomposition Method

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ABSTRACT

With the appliance of decomposition method evaluation of some integrals not expressible in the form of elementary functions nor adequately tabulated is performed in this paper.With the application of Ramanujan' s integral formula Laplace transform of the integrand of β-function is calculated conforming the known result.In this paper, various examples are taken to evaluate the integral using the method of asymptotic decomposition.

Keywords—Linear equations,convergent series, asymptotic decomposition.

INTRODUCTION

Evaluation of some difficult special integrals by reformulating them in the form of linear differential equation is performed in the form of easily convergent series with the help of asymptotic decomposition method [1-4] of Adomian solution of reduced differential equation of the integrations are easily,precisely and elegantly can be achieved.To illustrate this idea,we consider a more general Bernoulli's differential equation.

$\frac{dy}{dx}$ +P(x)y=Q(x)y^n ,y(0)=0 ----------(1)

which by substitution y^{-n+1}=v can be reduced easily to a linear differential equation of the form

$\frac{dv}{dx}$+(1-n)P(x)v=(1-n)Q(x) ,v(0)=0 ----------(2)

Likewise analysis of Adomian [1-3],equation (2) can be expressed in the form

Lv+P*v=Q^*,v(0)=0 ----------(3)

Where L≡ $\frac{d}{dx}$ P^*(x)=(1-n)P(x) , Q^*(x)=(1-n)Q(x)

The solution of (3) is given by

v(x)f(x)=$\int Q^*$ f(x)dx ----------(4)

where f(x)= exp($\int$ P^* dx) is the integrating factor of equation (3).

ANALYSIS

Adoption of asymptotic decomposition method for the right hand side of (4) is convenient to take the solution in the form

v=$\sum_{i=0}^{i=\infty} v_i$ ……....(5)

where v_{i+1} =(-1)$^{i+1}$ L^{i+1} (Q^*/P^*)/$P^{*\,i+1}$, where i=0,1,2,3,……..and v_0 =Q^*/P^* ……..(6)

Taking $P^* = -n$ and $Q^* = \phi(x)$, Ramanujan's integral formula is obtained like [8] as

$\int \phi(x)e^{-nx}\, dx = -e^{-nx} [\phi(x)/n + \phi'(x)/n^2 + \phi''(x)/n^3 + \ldots$ ……………………………………….(7)

The components v_i 's of the series solution are also can be obtained from the following as [2]

$v_0 = v(0) + L^{-1} Q^*$, $v_{i+1} = -L^{-1}(P^* v_i)$, i=0,1,2,3,…….. ……………………………………..(8)

(A) Illustration for use of Adomian's decomposition method.

(i) For a particular case, letting $P^*(x) = -n$, $Q^*(x) = x^m$ weobtain $v_0, v_1, v_2, \ldots$ using (8) and taking v(0)=1 as

$v_0 = v(0) + L^{-1} Q^* = 1 + \int_0^x x^m\, dx = 1 + x^{m+1}/(m+1)$

$v_1 = -L^{-1}(P^* v_0) = n \int_0^x (1 + x^{m+1}/(m+1))dx = n\{x + (x^{m+2}/((m+1)(m+2)))\}$

$v_2 = -L^{-1}(P^* v_1) = n^2\{x^2/2 + (x^{m+3}/((m+1)(m+2)(m+3)))\}$

………………………………………………………

……………………………………………………………………………………………(9)

and we get the solution

$v = \sum_{i=0}^{i=\infty} v_i = v_0 + v_1 + v_2 + v_3 + \ldots = 1 + nx + n^2 x^2/2!$

$+ n^3x^3/3! + \ldots + (x^{m+1}/(m+1))\{(1 + nx/(m+2) + n^2x^2/((m+2)(m+3)) + \ldots\}$

……………………………………………………………………………………………..(10)

as a special case m=0 gives the solution

$v = e^{nx} + (e^{nx} - 1)/n = \{1 + (1/n)\}e^{nx} - 1/n$

i.e., $y = \{(1 + (1/n))e^{nx} - (1/n)\}^{(1/(1-n))}$ ……………………………………………………….(11)

which is the corresponding exact solution.

(ii) Consider a second order linear differential equation

$v'' - 2v' + v = 0, v(0) = 1 , v'(0) = 1$ ……………………………………………………..(12)

where $v' = \frac{dv}{dx}$

without assuming a trial solution this equation can be written in the operator form as

$Lv + P^*v = Q^*$ where $L \equiv d^2/dx^2$, $P^* = -2d/dx + 1$, $Q^* = 0$, $v(0) = 1, v'(0) = 1$

Now applying Adomian's decomposition method

$v_0 = v(0) + L^{-1} Q^* = 1 + \iint_0^x 0\, dx = 1 + cx$

$v_1 = -L^{-1}(P^* v_0) = -L^{-1}\{(-2\frac{d}{dx} + 1)(1 + cx)\} = -L^{-1}(-2c + 1 + cx) = (2c-1)x^2/2! - cx^3/3!$

$v_2 = -L^{-1}(P^* v_1) = -L^{-1}[(-2\frac{d}{dx} + 1)\{(2c-1)x^2/2! - cx^3/3!\}] = 2(2c-1)x^3/3! - (4c-1)x^4/4! + cx^5/5!$

$v_3 = -L^{-1}(P^* v_2) = -L^{-1}[(-2\frac{d}{dx} + 1)\{2(2c-1)x^3/3! - (4c-1)x^4/4! + cx^5/5!\}]$

$= 4(2c-1)x^4/4! - 4(3c-1)x^5/5! + (6c-1)x^6/6! - cx^7/7!$

$v_4 = -L^{-1}(P^* v_3) = -L^{-1}[(-2\frac{d}{dx} + 1)\{4(2c-1)x^4/4! - 4(3c-1)x^5/5! + (6c-1)x^6/6! - cx^7/7!\}]$

$= 8(2c-1)x^5/5! - 4(8c-3)x^6/6! + 6(4c-1)x^7/7! - (8c-1)x^8/8! - cx^9/9!$ ……………………………..(13)

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Hence v= $\sum_{i=0}^{i=\infty} v_i$ =v_0+v_1+v_2+v_3+……=1+cx+(2c-1)x^2/2!-cx^3/3!+2(2c-1)x^3/3!-(4c-1)x^4/4! +cx^5/5!+4(2c-1)x^4/4!-4(3c-1)x^5/5!+(6c-1)x^6/6!-cx^7/7!+8(2c-1)x^5/5!-4(8c-3)x^6/6!+6(4c-1)x^7/7!-(8c-1)x^8/8!-cx^9/9! ……..(14)

Since v(0)=1,v′(0)=1 we have c=1 and hence
v=1+x+x^2/2!+x^3/3! +x^4/4! +……..= e^x ……(15)

Obviously, the solution of Bernoulli's equation is given by

y^{-n+1} = e^x i.e., y=$e^{x/(1-n)}$ ……(16)

(B) Illustration for use of asymptotic decomposition method

(i) Taking $P^*(x)$ = -x ,$Q^*(x)$=x^n in (3) we obtain from equation (4) as

$\int e\exp(-x^2/2)x^n\,dx = \exp(-x^2/2)\sum_{i=0}^{i=\infty} v_i$ …… ……..(17)

where v_0 =Q^*/P^* =x^n/(-x)= -x^{n-1}
v_1 = -(1/P^*)Lv_0 =(1/x)L(-x^{n-1}) = -(n-1)x^{n-3}

v_2 = -(1/P^*)Lv_1 =(1/x)L{-(n-1)x^{n-3} } = -(n-1)(n-3)x^{n-5}

v_3 =-(1/P^*)Lv_2 =(1/x)L{-(n-1)(n-3)x^{n-5} } = -(n-1)(n-3)(n-5)x^{n-7}

……..

v_k= -(n-1)(n-3)…….{n-(2k-1)}$x^{n-(2k+1)}$ ……..(18)

Evaluating the integral from x=0 to x=∞ *like*wise [8] equation (17) reduces to

$\int e\exp(-x^2/2)x^n\,dx$ = -[exp(-x^2/2){ x^{n-1} +(n-1)x^{n-3} +(n-1)(n-3)x^{n-5} +

+(n-1)(n-3)(n-5)x^{n-7} +……+(n-1)(n-3)(n-5)…….4.2] to be evaluated

from x=0 to x=∞ ... ...…….....(19)

=(n-1)(n-3)(n-5)…….4.2 = 2.1.2.2.2.3.2.4……2.(n-1)/2

=$2^{(n-1)/2}$ {1.2.3…….(n-1)/2} =$2^{(n-1)/2}$ ϒ($\frac{n+1}{2}$) ……(20)

As a particular case when n=0

$\int e\exp(-x^2/2)dx$ =$2^{-1/2}$ ϒ($\frac{1}{2}$) = $\sqrt{(\pi/2)}$

i.e., $\int_{-\infty}^{\infty} e\exp(-x^2/2)\,dx$ =2 $\int_0^{\infty} e^{-x*x/2})dx$ =$\sqrt{(2\pi)}$

i.e., (1/ $\sqrt{2\pi}$) $\int_{-\infty}^{\infty} e\exp(-x^2/2)dx$ =1 ……(21)

a known property of probability density function P(x)=(1/$\sqrt{(2\pi)}$)exp(-x^2/2) ,
-∞ < x < ∞ for standard normal distribution.

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(ii) Let $P^*(x) = -1/x^n$, $Q^*(x) = 1$ in equation (3) and so

$v_0 = Q^*/P^* = -x^n$

$v_1 = -(1/P^*)Lv_0 = x^n L(-x^n) = -nx^{2n-1}$

$v_2 = -(1/P^*)Lv_1 = x^n L\{-nx^{2n-1}\} = -n(2n-1)x^{3n-2}$

$v_3 = -(1/P^*)Lv_2 = x^n L\{-n(2n-1)x^{3n-2}\} = -n(2n-1)(3n-2)x^{4n-3}$

……..

$v_k = -n(2n-1)(3n-2)……..\{kn-(k-1)\}x^{(k+1)n-k}$, k=1,2,3,….. ……(22)

Here integrating factor $f(x)= \exp\{\int P^*(x)dx\} =\exp\{-(x^{-n+1}/(-n+1))\}$

Hence $\int_0^1 exp\{-(x^{-n+1}/(-n+1)\}dx= \sum_{i=0}^{i=\infty} exp\ \{-(x^{-n+1}/(-n+1))\}v_i$

$= -\sum_{i=0}^{i=\infty} exp\{-(x^{-n+1}/(-n+1))\}x^n\{1+nx^{n-1} +n(2n-1)x^{2n-2} +n(2n-1)(3n-2)x^{3n-3}$

$+……\}$

Its value = (value at upper limit x=1)-(value at lower limit x=0)

$= \exp\{1/(n-1)\} \sum_{r=1}^{r=\infty} n(2n-1)(3n-2)……..\{(r-1)n-(r-2)\}$ ……..,…(23)

(iii) Let $P^*(x)= -1/(2\sigma^2)$, $Q^*(x)= x^m$ in equation (3).

So $f(x)=\exp\{ \int P^*(x)dx\} = \exp\{-x/(2\sigma^2)\}$,

$v_0 = Q^*/P^* = -2\sigma^2x^m$

$v_1 = -(1/P^*)Lv_0 =2\ \sigma^2L(-2\sigma^2x^m) = -\ (2\sigma^2)^2\ mx^{m-1}$

$v_2 = -(1/P^*)Lv_1 = -(2\sigma^2)^3\ m(m-1)x^{m-2}$

…….

$v_k = -(2\sigma^2)^{k+1}\ m(m-1)……..(m-k+1)x^{m-k}$, k=1,2,3,……...(24)

Following the same procedure we have, $\int_0^\infty \exp(-x^2/2)x^m\ dx$

$= -2\sigma^2\ [\exp\{-x/(2\sigma^2)\}\{x^m +2\sigma^2mx^{m-1} +(2\sigma^2)^2m(m-1)x^{m-2} + …+(2\sigma^2)^m m!\}]$

When evaluated between the limits gives the value=$(2\sigma^2)^{m+1}\Gamma(m+1)$……(25)

Now put m= -1/2. The equation (25) reduces to

$\int_0^\infty exp(-x^2/2)x^{-1/2}\ dx=(2\sigma^2\ )^{-1/2+1}\ \Gamma(\ -\frac{1}{2}+1)=\sqrt{2}\sigma\sqrt{\pi}\ =\sqrt{2\pi}\sigma$ ……..(26)

This result conforms to the value of Rayleigh integral.

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Laplace transform of integrand of β-function using Ramanujan's formula

We rewrite Ramanujan's formula in equation (7) as

$\int f(x)e^{-sx}\,dx = -e^{-sx}[f/s + f'/s^2 + f''/s^3 + \ldots\ldots\ldots\ldots]$

Now Laplace transform of $f(x)= x^{m-1}(1-x)^{n-1}$,$0< x< 1$

$=0$, elsewhere

is $f(s) = \int_0^{\infty} e^{-sx} f(x)dx = \int_0^1 e^{-sx} x^{m-1}(1-x)^{n-1}\,dx$

$= e^{-sx}\{f/s + f'/s^2 + f''/s^3 + \ldots\ldots\ldots\ldots.,\}$

$= -e^{-sx}\{(x^{m-1}(1-x)^{n-1})/s + (x^{m-2}(1-x)^{n-2}(m-1-(m+n-2)x))/s^2 + \ldots\ldots\ldots$

$+(m-1)(m-2)\ldots\ldots.(m-n+1)x^{m-n}(-1)^n(n-1)!/s^n + \ldots\ldots.+$

$(m-1)!(1-x)^{n-1}/s^m + \ldots\ldots\ldots\ldots\}$

$=(-1)^{n-1}(n-1)!(m-1)(m-2)\ldots\ldots\ldots..(m-n+1)e^{-s}/s^n + (m-1)!/s^m$

$=(m-1)!\,[(-1)^{n-1}(n-1)!\,e^{-s}/((m-n)!s^n) + 1/s^m]$ ……(27)

First part of equation (27) vanishes as $e^{-sx}\rightarrow 0$ when $x\rightarrow\infty$.

For a particular case when n=1 and we consider the limit from 0 to ∞ *the* Laplace

transform of x^{m-1} is obtained as $L[x^{m-1}]=(m-1)!/s^m$…….....l.(28)

from equation (27) and it conforms to the known result.

CONCLUSION

Application of asymptotic decomposition method is quite useful to evaluate complicated integrals in the form of rapidly convergent series by transforming it into a linear differential equation.Moreover, Laplace transform of functions can be directly achieved through the use of Ramanujan's integral formula which is nothing but a sequence of solution of a linear differential equation of first order and first degree.Actually,use of asymptotic decomposition method as well as Ramanujan's integral formula yield solution regarding evaluation of various complicated integrals.

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