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Lattices of Reduction and Subset-Induced Topologies

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Abstract: Induced topologies have been studied only from the standpoint of a superset down to its subset to get what we call subspace topology. Here we turn the focus around and show that subsets can induce topologies on their supersets. Also, induced topologies have so far only been constructed by collecting the intersections of open sets of a superset with a subset. Here again we extend the focus and show that a superset will always induce topologies on their subsets through other means than by taking intersections of open sets with a subset. All these warrant further research into a more extensive and comprehensive study of induced topologies; to establish how some topological properties such as compactness, separation axioms, etc. are shared or inherited in the wider context of inducement of topologies. The concept of reducible topologies has been explored and published by the authors before [1]. Here we extend the research by proving that any pairwise comparable family F of subsets of a set X generates a reducible topology τ on X, and that the chain C of reductions of τ can be constructed in such a way that card(F) = card(C).

Keywords: Induced Topology, Reduction of Topology, Lattice of Reductions, Chain of Reductions, Comparison of Topologies
Mathematics Subjects Classification (MSC) 2020: 54A05, 54A10

Reducible Topologies—Lattices

I. Introduction

We recall the following definitions.

Definition 1.1 A relation R on a set X is called a partial order on X if

1. R is reflexive; in that xRx, for all $x \in X$,
2. R is transitive; in that xRy and yRz implies xRz,
3. R is anti-symmetric; in that xRy and yRx implies $x = y$.

Definition 1.2 A set X on which a partial order is defined is called a partially ordered set; in brief, a poset.

Definition 1.3 If X is a poset, with partial order R, and xRy, then we say that x precedes y, written $x \prec y$. We then analogously also say that y dominates x. If x precedes y and $x \neq y$, we say that x properly precedes y, or y properly dominates x.

Definition 1.4 Let X be a poset with R. Then x is called a lower bound of y if $x \prec y$; and then y is called an upper bound of x.

Definition 1.5 Let X be a poset with R. An element x_0 of X is called the first or the least element of X if x_0 precedes every other element of X. The last or greatest element of X is that which dominates every other element of X.

Definition 1.6 Let X be a poset. An element x_0 of X is called a minimal element if no element of X properly precedes x_0.

If x_0 is a minimal element of a poset X and $x \prec x_0$, then $x = x_0$. Also, every first element is a minimal element but a minimal element may not be a first element.

Definition 1.7 Let X be a poset. An element y_0 of X is called a maximal element if no element of X properly dominates y_0.

Definition 1.8 Let X be a poset. Let T be a subset of X. A lower bound of T is an element of X which precedes every element of T. The greatest lower bound (g.l.b.) of T is the lower bound which dominates every other lower bound of T. The g.l.b. of T is also called the infimum of T, and denoted inf(T).

Definition 1.9 Let X be a poset and let T be a subset of X. An upper bound of T is an element of X which dominates every element of T. The least upper bound (l.u.b.) of T is the upper bound which precedes every other upper bound of T. The l.u.b. of T is also called the supremum of T, and denoted sup(T).

Definition 1.10 Two elements x, y of a poset X are said to be comparable if either $x \prec y$ or $y \prec x$.

Definition 1.11 A lattice is a poset in which every two elements have a g.l.b and an l.u.b.

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II. Development of Lattice of Topologies

Let $C = \{\tau_\alpha : \alpha \in \Delta\}$ be a chain of reductions of a topology τ on a set X. Then C, with the relation of set inclusion $\subseteq$ is a poset. We also see that C is totally ordered. If τ_{α_1} and τ_{α_2} are two topologies in C such that, say, τ_{α_1} is weaker than τ_{α_2}, then the g.l.b. of the sub-family $T = \{\tau_{\alpha_1}, \tau_{\alpha_2}\}$ of C, that is, inf(T), is τ_{α_1}. Also sup(T) = τ_{α_2}. Hence C is a lattice of topologies by set inclusion.

Let R be another relation on the chain C, where $\tau_\alpha R \tau_r$ if $\tau_\alpha \leq \tau_r$. That is, the relation R ($\leq$) on C, now, is that of comparison of topologies. With this relation on C, we see again that C is a lattice of topologies. What we have just established is the following.

Corollary 1.12 Every chain C of reductions of a topology on a set X is a lattice in at least two ways.

Observations

Every set on which a partial order is defined is not a lattice; that is, not every poset that is a lattice. In particular, every family of topologies is not a lattice. For example, if the topologies in a family F are not comparable, then the family F would not be a lattice in either of the ways; but F would still be a poset in the two ways.

If a family of subsets of a set X is pairwise comparable by set inclusion (i.e. totally ordered by set inclusion), then it generates a topology (on X) which has a chain of reductions. This indeed is a theorem which marks the end and climax of this section.

Theorem 1.1 Any (set inclusion) pairwise comparable family F of subsets of a set X generates a reducible topology τ on X. And the chain C of reductions of τ can be constructed in such a way that card(F) = card(C).

Proof: Let $F = \{A_\alpha : A_\alpha \subset X\}_{\alpha\in\Delta}$ be a family of (set inclusion) pairwise comparable subsets of X. Let $A_{\alpha 1}$ and $A_{\alpha 2}$ be two elements of F such that, say, $A_{\alpha 1} \subset A_{\alpha 2}$. Let $\gamma_1 = A_{\alpha 1}$-induced topology on X and $\gamma_2 = A_{\alpha 2}$-induced topology on X. If γ_1 and γ_2 are not comparable, let $\tau_1 = \gamma_1$ and $\tau_2 = \gamma_1 \nabla \gamma_2$, the join of γ_1 and γ_2 (defined as the weakest topology, on X, finer than both γ_1 and γ_2). Then τ_1 and τ_2 are two comparable topologies on X. Precisely, τ_1 is strictly weaker than τ_2.

Since F is pairwise comparable, the sets in F can be arranged such that

$$A_\alpha \subset A_r \subset \cdots.$$

It follows from the construction above that these sets in F have, corresponding to them, a family $C = \{\tau_\alpha\}_{\alpha\in\Delta}$ of topologies on X, which is pairwise comparable in that

$$\tau_\alpha \leq \tau_r \leq \cdots.$$

It is easy to see that C is equivalent to F; that is, card(C) = card(F).

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It is easier to see the existence of the chain C, constructed in the proof of the theorem if we remember that the construction can actually be done through inducement by the discrete topologies of $A_{\alpha 1}$ and $A_{\alpha 2}$; or, by what is similar, first getting a topology on $A_{\alpha 2}$ and then using this to induce a topology on $A_{\alpha 1}$; and then finally using these two topologies to construct subset-induced topologies on X.

III. Subset-induced Topologies

Proposition 2.1 If $X \subset E$, then any topology, say τ_X, on X induces a topology, say τ_{XE}, on E, given by $\tau_X(E) = \tau_X \cup \{E\}$.

Proof: It is easy to see that $\emptyset \in \tau_X(E)$, since $\emptyset \in \tau_X$. Also $E \in \tau_X(E)$, by definition. Let $\{G_i: i = 1, \ldots, n\}$ be a sub-collection of $\tau_X(E)$. We show that the intersection $\bigcap_{i=1}^n G_i$ belongs to $\tau_X(E)$. Clearly $\bigcap_{i=1}^n G_i \in \tau_X(E)$ if any of the G_i comes from τ_X. If all the G_i are each equal to E, then $E = \bigcap_{i=1}^n G_i$ is an element of $\tau_X(E)$. Hence in any case $\tau_X(E)$ is closed under finite intersections. Let $\{G_\alpha: \alpha \in \Delta\}$ be any family of sets in $\tau_X(E)$. If one of these sets equals E, then their union would equal E, which belongs to $\tau_X(E)$. If none of these sets equals E, then each of them belongs to τ_X and hence their union belongs to τ_X which is itself a subfamily of $\tau_X(E)$. These imply that $\tau_X(E)$ is also closed under arbitrary unions, and is therefore a topology on E.

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Definition 2.1 The topology $\tau_X(E)$, on E, is called an X-topology on E; or a topology induced on E by the topology τ_X on X.

Observe that one subset can induce several topologies on its superset.

Proposition 2.2 Let (E,τ) be a topological space, and let $X \in \tau$ be a τ-open subset of E. Let $\tau_X = \{G \in \tau : G \subset X\}$. Then τ_X is a topology on X.

Proof:

1. $\emptyset \in \tau_X$, since $\emptyset \in \tau$ and $\emptyset \subset X$.
2. $X \in \tau_X$, since $X \in \tau$ and $X \subset X$.

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3. Let $\{G_i\}_{1\leq i\leq n} \subset \tau_X$ be any finite number of sets of τ_X; and let $N = \bigcap_{i=1}^{n} G_i$

   be the intersection of these sets. Then clearly $N \in \tau$, as the intersection of a finite number of sets of τ. Also it is clear that $N \subset X$, since it is the intersection of some subsets of X. Hence $N \in \tau_X$.

4. Let $\{G_\alpha\}_{\alpha\in\Delta} \subset \tau_X$ be any family of sets of τ_X. Then $\bigcup_{\alpha\in\Delta} G_\alpha = U \in \tau$,

since τ is closed under arbitrary unions. Also $U \subset X$, as a union of subsets of X. Hence $U \in \tau_X$, implying that τ_X is closed under arbitrary unions and, hence, a topology on X.

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Definition 2.2 With X, E and τ_X as given in proposition 2.2, let $\tau_X(E) = \tau_X \cup \{E\}$ be an X-topology on E. Then $\tau_X(E)$ is an open subset induced topology on E.

IV. Discussions

1. Proposition 2.2 shows that a superset can induce a topology on its subset through other means than by collecting the intersections of the subset with the open sets of the superset. But someone might still say (against the idea of Proposition 2.2) that since X is open in E, the induced topology τ_X on X is the same thing as what would have resulted if the intersections of X with open sets of E were collected. To properly see the difference between the old method and the new method of inducement here, see proposition 2.3 below; it is the general form of proposition 2.2.
2. Definition 2.2 shows that a subspace topology can in turn induce a topology on its 'superspace', and that such a subset-induced topology on a superset may actually be comparable with the original topology of the superset.

Proposition 2.3 Let (E, τ) be a topological space, and let X be any subset of E. Let $\tau_X = \{G \in \tau : G \subset X\} \cup \{X\}$. Then τ_X is a topology on X.

Application

Let (R, u) denote the usual topological space of the set of real numbers, and let X = [a, b] be a closed interval in R. Using proposition 2.3, the u-induced topology on X is the family $\tau_X = \{G \in U : G \subset X\} \cup \{X\}$. It is clear that no set of the form [a, c) or (c, b] is open in this induced topology of X, where $a < c < b$. It is also clear that all such half-open intervals are open in X when the inducement is done using the old method of collecting the intersections of X with the open sets of (R, u).

V. Summary and Conclusion

1. Induced topologies have been studied only from the standpoint of a superset down to its subset. We showed that subsets can induce topologies on their supersets.
2. Induced topologies have so far only been constructed by collecting the intersections of open sets of a superset with a subset [5], [6], [9], [10], and [13]. We expanded the focus and showed that a superset can always induce topologies on their subsets through other means than by taking intersections of open sets with the subset.
3. Further extensive and comprehensive study of induced topologies needs to be done to establish how some topological properties such as compactness, separation axioms, etc. are shared in the wider context of inducement of topologies.
4. We proved that any pairwise comparable family F of subsets of a set X generates a reducible topology τ on X, and that the chain C of reductions of τ can be constructed in such a way that the cardinality of F equals the cardinality of C.

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