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# Reducibility of Topologies

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**Abstract:** The concepts of *weak* or *strong* reduction of topologies are introduced. Closely related to these, and introduced as well, are the concepts of *weak* and *strong* base reduction of topologies. We also defined *extensible topologies*; and defined *weak* and *strong* base extension of topologies. We proved that there exists a topology $\gamma$, weaker than a weak topology $\tau$, on $X$, which has a chain of strong reductions if one of the range spaces, say $(X_\alpha, \tau_\alpha)$ of $\tau$, has a chain of strong reductions. It is proved that the usual topology of the set R of real numbers can be reduced in the weak sense to chains of infinite families of pairwise comparable topologies; and that the usual topology of R can neither be reduced in the normal sense nor in the strong sense. We proved that a weak topology has a chain of weaker topologies if one of its range topologies is reducible to a chain of topologies.

**Keywords:** Reduction of Topology, Strong, Normal and Weak Reduction of Topologies, Extension of Topologies, Weak Topology, Comparability of Topologies, Base Reduction of Topologies Mathematics Subjects Classification (MSC) 2020: 54A05, 54A10

## I. Introduction

Throughout, $X$ is a nonempty set.

**Definition 1.1** *A topology $\tau$ on $X$ is said to be* **strongly reducible** *or reducible in the* **strong sense** *if $\exists G \in \tau$ such that $\tau_1 = \tau - \{G\}$ is a topology on X. The topology $\tau_1$ is called a strong reduction of $\tau$.*

### Example 1

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Then $\tau$ on $X$ is strongly reducible, since there exists $\{a\} \in \tau$ such that $\tau_1 = \tau - \{\{a\}\} \equiv \{\emptyset, X, \{c\}, \{a, c\}\}$ is a topology on $X$. Conversely, $\tau_1$ is a strong reduction of $\tau$.

Let $X = \{a, b, c\}$ and $\tau = 2^X = \{\emptyset, X, \{a\},\{b\},\{c\},\{a ,b\},\{a, c\},\{b, c\}\}$. Then $\tau = 2^X$ is not strongly reducible.

**Definition 1.2** *A topology $\tau$ on $X$ is said to be* **normally reducible** *or simply reducible, or* **reducible in the normal sense** *if there exist $G_i \in \tau$ $(i = 1,\cdots,m)$; $m \in$ IN such that $\tau_1 = \tau - \{G_1,\cdots,G_m\}$ is a topology on X. Such a topology $\tau_1$ is called a normal reduction of $\tau$, or simply a reduction of $\tau$.*

### Example 2

Let $X = \{a, b, c\}$ and $\tau = 2^X = \{\emptyset, X,\{a\},\{b\},\{c\},\{a, b\},\{a, c\},\{b, c\}\}$. Then $\tau = 2^X$ is normally reducible, to $\tau_1 = \tau - \{\{c\},\{b, c\}\} \equiv \{\emptyset, X,\{a\},\{b\},\{a, b\},\{a, c\}\}$.

**Definition 1.3** *A topology $\tau$ on $X$ is said to be* **weakly reducible** *or reducible in a weak sense if* there exist$\{G_\alpha \in \tau : \alpha \in \Delta\}$ *such that $\tau_1 = \tau - \{G_\alpha \in \tau : \alpha \in \Delta\}$ is a topology on X. The topology $\tau_1$ is called a weak reduction of $\tau$.*

### Example 3

Let (IR, $U$) denote set *R of real numbers* with its usual topology $U$. Let $X = (-\infty,0)$, and $\tau_X = \{G \in U : G \subset X\} \cup \{\text{IR}\}$. Then $\tau_X$ is a weak reduction of $U$, since $\tau_X = U - \{G \in U : G$ is not a subset of $X\}$.

### Remark

1. Strongly Reducible =⇒ Normally Reducible =⇒ Weakly Reducible. But the converses are not always true.
2. The indiscrete topology of a set cannot be reduced in any sense (strong, normal or weak). In fact, it is the weakest reduction of any topology.
3. In the first two examples above we saw that the discrete topology of $X$ is not reducible in the strong sense. This is actually a general fact for the discrete topology of any set $X$ whose cardinality is greater than 2; and we state and prove that below as a theorem.
4. The discrete topology is not the only topology that is irreducible in the strong sense. The usual topology of IR is not reducible in the strong sense. This is stated and proved below as a proposition.

**Theorem 1.1** *(a) The discrete topology of X cannot be reduced in the strong sense if the cardinality of X is greater than 2. (b) Every non-indiscrete topology on a set X can be reduced in some sense (strong, normal or weak).*

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**Proof:**

(a) Let the cardinality of $X$ be greater than 2 and let $(X, D)$ be a discrete topological space. Suppose $G \in D$ and $\eta = D - \{G\}$. We need to show that $\eta$ is not a topology on $X$.

Without loss of generality, suppose $G \neq \{a\}$. Then there exist at least two proper subsets of $G$ and each is in $D$ (as the discrete topology) and hence separately in $\eta$. Since $G$ is the union of all the proper subsets of $G$, it follows (as $G \notin \eta$) that $\eta$ is not closed under arbitrary unions and is hence not a topology on $X$.

Now suppose $G = \{a\}$, a singleton. Then from hypothesis $X$ contains two other mutually distinct elements $x_1$, $x_2$, each different from $a$. The sets $G_1 = \{a, x_1\}$ and $G_2 = \{a, x_2\}$ are in $D$ (as the discrete topology) and hence in $\eta$. It is easy to see that $G_1 \cap G_2 = G \notin \eta$; hence $\eta$ is not a topology on $X$.

(b) Let $\tau$ be a non-indiscrete topology on $X$. Then the indiscrete topology $\{\emptyset, X\}$ on $X$ is a reduction of $\tau$ in some sense. The proof is complete.

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**Proposition 1.1** *The usual topology $U$ of the set* R *of real numbers is not reducible in the strong sense.*

**Proof:**

Let (R,$U$) denote IR with its usual topology. Let $\eta = U - \{(a\ ,b)\}$,for some

$(a, b) \in U$. We show that $\eta$ is not a topology on IR. For each $n \in N$ let

$$G_n = \left(a + \tfrac{b-a}{2n}\ , b - \tfrac{b-a}{2n}\right).$$

Then each $G_n$ is an element of $U$ and an element of $\eta$. Clearly

$$(a, b) = \bigcup_{n=1}^{\infty} G_n$$

and since $(a, b) \notin \eta$ it follows that $\eta$ is not closed under

arbitrary unions and is hence not a topology on IR.

■

**Note**

- Not only that the usual topology of R cannot be reduced in the strong sense; it can also not be reduced in the 'normal' sense.
- There can be found many other topologies which are not reducible in the strong sense. For example the lower limit topology of R is not strongly reducible and the upper limit topology of R is not strongly reducible. Yet infinitely many topologies can be reduced in the strong sense—for example, the discrete topology of any set with only two elements has a chain of strong reductions.
- So far, it may appear that the only examples of strongly reducible topologies available are finite topologies or topologies on finite sets. Infinite topologies and indeed topologies on infinite sets can be strongly reducible. The next example illustrates this.

**Example 4**

Let N= $\{0,1,2,\cdots\}$ denote the set of natural numbers. For each $n \in$ N let $G_n$ be the set of all real numbers *excluding* the first $n$ natural numbers. Thus for instance

$G_0 =$ R$- \{\} =$ R;

$G_1 =$ R$- \{0\}$;

$G_2 =$ R$- \{0, 1\}$;

$G_3 =$ R$- \{0, 1, 2\}$;

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$G_n = R - \{0, 1, 2, 3, \cdots, n-1\}$

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Let $T_{CN} = \{\emptyset, G_n\}_{n \in N}$. Then it is easy to see that

1. The empty set is in $T_{CN}$, from the way $T_{CN}$ is defined.
2. The whole set R of real numbers is in $T_{CN}$.
3. $T_{CIN}$ is closed under finite intersections.
4. And that $T_{CIN}$ is closed under arbitrary unions.

Hence $T_{CIN}$ is a topology on IR. We see that $T_{CIN}$ is strongly reducible since, say $\tau = T_{CIN} - \{G_5\}$ is a topology on IR. (The topology $T_{CIN}$ here is one of our interesting constructions in this work.)

**Definition 1.4** *A topology $\tau$ on X, with base B, is said to be* ***strongly base reducible*** *or base reducible in the strong sense if there exists $B_0 \in B$ such that $B_1 = B - \{B_0\}$ is a base for a topology $\tau_1$ on X strictly coarser than $\tau$. Such a topology $\tau_1$ is called a strong base reduction of $\tau$.*

**Example 5**

Let $X = \{a ,b, c\}$ and $\tau_1$ on $X$ be $\tau_1 = 2^X = \{\emptyset, X, \{a\},\{b\},\{c\},\{a, b\},\{a, c\},\{b, c\}\}$.

Let $B_1 = \{\{a\},\{b\},\{c\}\}$ be a base for the topology $\tau_1$ on $X$. Then $\tau_1$ with the base $B_1$ is not strongly base reducible.

However, if we endow $X$ with the topology $\tau_2 = \{\emptyset, X, \{a\},\{b\},\{a, b\},\{a, c\}\}$, with base $B_2 = \{\{a\},\{b\},\{a, c\}\}$, then $\tau_2$ would be strongly base reducible, for there exists $\{a\} \in B_2$ such that $B_3 = B_2 - \{\{a\}\} \equiv \{\{b\},\{a, c\}\}$ is a base for a topology $\tau_3$ on $X$ given by

$\tau_3 = \{\emptyset, X, \{b\},\{a, c\}\}$.

**Definition 1.5** *A topology $\tau$ on X, with base B, is said to be* ***base reducible*** *if there exists $B_i \in B$($i = 1,\cdots,m$; $m \in N$) such that $B_1 = B - \{B_i : i = 1,\cdots,m\}$ is a base for a topology $\tau_1$ on X strictly coarser than $\tau$. Such a topology $\tau_1$ is called a base reduction of $\tau$.*

**Definition 1.6** *A topology $\tau$ on X, with base B, is said to be* ***weakly base reducible*** *or base reducible in the weak sense if* there exist$\{B_\alpha \in B : \alpha \in \Delta\}$ *such that $B_1 = B - \{B_\alpha : \alpha \in \Delta\}$ is a base for a topology $\tau_1$ on X strictly coarser than $\tau$. Such a topology $\tau_1$ is called a weak base reduction of $\tau$.*

**Example 6**

Let (R, $U$) denote the usual topological space of R. Then $B = \{(a, b): a, b \in R\}$ is a base for $U$. Let $B_1 = \{B_\alpha \in B: B_\alpha \subset (-\infty, 0)\} \cup \{R\}$. Then $B_1$ is a base for a topology on IR (namely the topology $\tau_X = \{G \in U: G \subset X\} \cup \{R\}$ given after Definition 1.3) strictly weaker than $U$. That is, the topology $\tau_X$ is a weak base reduction of (R, $U$).

**Remark**

A strongly base reducible topology is base reducible. A base reducible topology is weakly base reducible but converses of these do not hold in general.

**Definition 1.7** *A topology $\tau$ on X is said to be*

1. ***strongly extensible*** *if $\exists G \subset X$, $G \notin \tau$ such that $\gamma = \tau \cup \{G\}$ is a topology on X. The topology $\gamma$ is then called a strong extension of $\tau$;*
2. ***extensible*** *if $\exists \{G_i \subset X : G_i \notin \tau;\ i = 1,\cdots,m;\ m \in N\}$ such that $\gamma = \tau \cup \{G_1,\cdots,G_m\}$ is a topology on X. The topology $\gamma$ is called an extension of $\tau$;*
3. ***weakly extensible*** *if $\exists \{G_\alpha \subset X : G_\alpha \notin \tau; \alpha \in \Delta\}$ such that $\gamma = \tau \cup \{G_\alpha\}_{\alpha \in \Delta}$ is a topology on X. Such a $\gamma$ is then called a weak extension of $\tau$.*

**Definition 1.8** *A topology $\tau$ on X with base B is said to be*

1. ***strongly base extensible*** *if $\exists B_0 \subset X$, $B_0 \notin B$ such that $\Omega = B \cup \{B_0\}$ is a base for a topology $\gamma$ on X finer than $\tau$. The topology $\gamma$ is then called a strong base extension of $\tau$;*
2. ***base extensible*** *if $\exists \{B_i \subset X, B_i \notin B, i = 1,\cdots,m;\ m \in N\}$ such that $\Omega = B \cup \{B_i;\ i = 1,\cdots,m\}$ is a base for a topology $\gamma$ on X, finer than $\tau$. The topology $\gamma$ is called a base extension of $\tau$;*
3. ***weakly base extensible*** *if $\exists \{B_\alpha \subset X: B_\alpha \notin B, \alpha \in \Delta\}$ such that $\Omega = B \cup \{B_\alpha : \alpha \in \Delta\}$ is a base for a topology $\gamma$ on X finer than $\tau$. In this case the topology $\gamma$ is called a weak base extension of $\tau$.*

The following propositions hold true obviously from the definitions above.

**Proposition 1.2** *A topology $\tau$ on X is*

1. *strongly extensible if, and only if, $\tau$ is a strong reduction of some topology $\gamma$ on X;*

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2. *extensible if, and only if, $\tau$ is a reduction of some topology $\gamma$ on X;*
3. *weakly extensible if, and only if, $\tau$ is a weak reduction of some topology $\gamma$ on X.*

**Proposition 1.3** *A topology $\tau$ on X with base B is*

1. *strongly base extensible if, and only if, $\tau$ is a strong base reduction of some other topology $\gamma$ on X;*
2. *base extensible if, and only if, $\tau$ is a base reduction of some topology $\gamma$ on X;*
3. *weakly base extensible if, and only if, $\tau$ is a weak base reduction of some topology $\gamma$ on X.*

**Definition 1.9** *Let $\tau$ be a strongly reducible topology on X. If $\tau_1$ is a strong reduction of $\tau$, $\tau_2$ a strong reduction of $\tau_1$, $\tau_3$ a strong reduction of $\tau_2$, and so on, then the pairwise comparable family*

$$C = \{\tau n\}\ n \in \mathrm{N}$$

*of topologies on X is called a chain of strong reductions of $\tau$ on X.*

**Example 7**

Let $X = \{a, b, c\}$ and $\tau$ on $X$ be $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Then $\tau_1 = \{\emptyset, X, \{c\},\{a, c\}\}$ or $\tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}$ is a strong reduction of $\tau$. Also $\tau_2 = \{\emptyset, X, \{c\}\}$ or $\tau_2 = \{\emptyset, X, \{a\}\}$ or $\{\emptyset, X, \{a, c\}\}$ is a strong reduction of $\tau_1$.

And $\tau_3 = \{\emptyset, X\}$ is a strong reduction of $\tau_2$. Hence the family

$$C_1 = \{\tau_1, \tau_2, \tau_3\}$$

is a chain of strong reductions of $\tau$.

For the topology $\tau$ on $X$ given by $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ a chain of strong reductions can be obtained as follows: $\tau_1 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$; $\tau_2 = \{\emptyset, X,\{a\},\{a, c\}\}$; $\tau_3 = \{\emptyset, X,\{a\}\}$; and $\tau_4 = \{\emptyset, X\}$.

We see that

$$\tau_4 < \tau_3 < \tau_2 < \tau_1 < \tau;$$

and that

$$C_2 = \{\tau_1, \tau_2, \tau_3, \tau_4\}$$

is a chain of strong reductions of $\tau$.

**Remark**

We notice first that a strongly reducible topology can be reduced to a chain of pair-wise comparable topologies. Secondly, there is often more than one way of getting a chain of strong reductions of a strongly reducible topology.

The chains $C_1$ and $C_2$ in the last example are simple enough, in that they are (each) finite. Hence one may wonder if the only examples of chain of strong reductions (of a topology) that could be found are those that are finite. Actually examples of denumerable chains of reductions exist. For example, the topology $T_{CN}$ on R that we constructed above, just before definition 3.6, has a countably infinite chain of strong reductions. To see this, we observe that

$$T_{C\mathbb{N}} = \bigcup_{n=0}^{\infty} \{\tau_n\}$$

where $\tau_0 = \{\emptyset,\mathrm{IR}\}$, $\tau_1 = \tau_0 \cup \{G_1\}$, $\tau_2 = \tau_1 \cup \{G_2\}$, and so on. Then

$$C = \{\tau_0, \tau_1, \tau_2,\cdots\}$$

is a countably infinite family of strong reductions of $T_{CIN}$.

**Definition 1.10** *Let $\tau$ be a (strongly or weakly) reducible topology on X. If $C_1$ and $C_2$ are two chains of (weak or strong) reductions of $\tau$ such that for each $\tau_{1i} \in C_1$, there exists $\tau_{2j} \in C_2$ such that $\tau_{1i}$ is weaker than $\tau_{2j}$, then we say that the chain **$C_1$ is weaker than the chain $C_2$**.*

**Definition 1.11** *Let $\tau$ be a (strongly or weakly) reducible topology on X. If $C_1$ and $C_2$ are two chains of (weak or strong) reductions of $\tau$ such that for each $\tau_{1i} \in C_1$, there exists a $\tau_{2j} \in C_2$ such that $\tau_{1i}$ is strictly weaker than $\tau_{2j}$, then we say that the chain **$C_1$ is strictly weaker than the chain $C_2$**.*

**Definition 1.12** *If $C_1$ and $C_2$ are two chains of reductions of $\tau$ on X such that $C_1$ is weaker than $C_2$ and $C_2$ is weaker than $C_1$, then we say that **$C_1$ is equivalent to $C_2$**.*

**Definition 1.13** *If $C_1$ is not weaker than $C_2$ and $C_2$ is not weaker than $C_1$, then we say that **$C_1$ and $C_2$ are not comparable**.*

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**Example 8**

Let $X = \{a, b, c\}$ and $\tau$ on $X$ be $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Let $C_1 = \{\tau_{11}, \tau_{12}, \tau_{13}\}$ where $\tau_{11} = \{\emptyset, X, \{c\}, \{a, c\}\}$, $\tau_{12} = \{\emptyset, X, \{c\}\}$, and $\tau_{13} = \{\emptyset, X\}$. Then $C_1$ is a chain of strong reductions of $\tau$.

Let $C_2 = \{\tau_{21}, \tau_{22}, \tau_{23}\}$ where $\tau_{21} = \{\emptyset, X, \{a\}, \{a, c\}\}$, $\tau_{22} = \{\emptyset, X, \{a, c\}\}$, and $\tau_{23} = \{\emptyset, X\}$. Then $C_2$ is another chain of strong reductions of $\tau$.

We see that $C_1$ and $C_2$ are not comparable because the topology $\tau_{12}$ in $C_1$ is not comparable to any topology in $C_2$; and $\tau_{21}$ in $C_2$ is not comparable to any topology in $C_1$.

**Example 9**

Let $C_1$ remain as in the example above and let $C_3 = \{\tau_{31}, \tau_{32}, \tau_{33}\}$ where $\tau_{31} = \{\emptyset, X, \{c\}, \{a, c\}\}$, $\tau_{32} = \{\emptyset, X, \{a, c\}\}$, and $\tau_{33} = \{\emptyset, X\}$. Then $C_3$ is another chain of strong reductions of $\tau$ and we see that $C_1$ is weaker (but not strictly) than $C_3$, since every topology in $C_1$ is weaker than $\tau_{31}$. And if we also observe that every topology in $C_3$ is weaker than $\tau_{11}$, then we know that $C_1$ and $C_3$ are equivalent.

**Example 10**

Let (R, $u$) denote the set of real numbers with its usual topology. Let Z denote the set of integers. For each $z \in$ Z, let $X_z$ be the $u$-open interval $X_z = (-\infty, z)$. Then clearly

$$\{G \in u : G \subset X_z\} = \{G \in u : G \subset (-\infty, z)\}$$

is a topology on $X_z$. Let $\tau_z = X_z$-topology on R; in that $\tau_z = \{G \in u : G \subset X_z\} \cup \{R\} = \{G \in u : G \subset (-\infty, z)\} \cup \{R\}$.

Then clearly if $z_1 < z_2$, we have $X_{z_1} \subset X_{z_2}$ and $\tau_{z1}$ is weaker than $\tau_{z2}$. Hence the family

$$C_Z = \{\tau_z : z \in Z\}$$

is a chain of weak reductions of the usual topology on $R$, in that

$$\cdots < \tau z{-}2 < \tau z{-}1 < \tau z0 < \tau z1 < \tau z2 < \cdots < u,$$

where $u$ is the usual topology on R.

For each $n \in$ N (= the set of natural numbers), let $X_n = (-n, n)$ and let $\tau_n = \{G \in u : G \subset X_n\} \cup \{R\}$ be an $X_n$-topology of R, obtained from the usual topology on R. For instance, $X_1 = (-1,1)$ and $\tau_1 = \{G \in u : G \subset X_1\} \cup \{R\}$ is an $X_1$-topology on R strictly weaker than the usual topology on R. Also $X_2 = (-2, 2)$ and $\tau_2 = \{G \in u : G \subset X_2\} \cup \{R\}$ is an $X_2$-topology of R obtained from the usual topology on R. And so on. Then

$$C_N = \{\tau_n : n \in N\}$$

is a chain of weak reductions of $u$. Since, for each $n \in$ IN, the set $(-n, n)$ is a proper subset of $(-\infty, n)$, and we see that the chain

$$C_N = \{\tau_n : n \in N\}$$

is strictly weaker than the chain

$$C_Z = \{\tau_z : z \in Z\}.$$

What happens on a weak topology in terms of reducibility? We will now show that if $\tau$ is a weak topology on a set $X$, and one of the range spaces of $(X, \tau)$ is reducible in the strong sense, then there exists a chain of weak topologies, each weaker than $\tau$, on $X$ (generated by the fixed family of functions), which are a chain of reductions of $\tau$ (not necessarily in the strong sense) if the function associated with the strongly reducible range space has requisite properties. We prove this next in a theorem.

The following lemma will be useful in the theorem that follows after.

**Lemma 1.1** *If $\tau$ is a topology on $X$ and $\tau_1 = \tau \cup \{G\}$ is a topology on $X$*

*(where $G \notin \tau$), then $\tau_1$ is only one set, $G$, strictly finer than $\tau$.*

**Proof:**

$\tau_1$ is a strong extension of $\tau$ and is, hence, only one set strictly finer than $\tau$.

■

**Note**

What Lemma 1.1 says is that the introduction of just one set $G$ into a topology $\tau$ to produce another topology $\tau_1$ does not make $\tau_1$ to have more than one open set (either from finite intersections or arbitrary unions) than $\tau$—and that the extra open set is precisely $G$.

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**Theorem 1.2** *Let $(X, \tau)$ be a weak topological space generated by the family $\{(X_\alpha, \tau_\alpha)\}$ of topological spaces, together with the family $\{f_\alpha\}$ of functions. There exists a chain of weak topologies, each weaker than $\tau$, on $X$ (generated by this fixed family of functions), which are a chain of reductions of $\tau$ if (a) one of the range spaces, say $\tau_\alpha$, has a chain of strong reductions, (b) $f_\alpha$ is one-to-one, and (c) $f_\alpha$ maps into all the elements of each topology in the chain of strong reductions of $\tau_\alpha$.*

**Proof:**

Let $(X_\alpha, \tau_\alpha)$ be the range space meeting the hypotheses, for some $\alpha \in \Delta$, and let

$$C_\varphi = \{\tau_r : r \in \varphi\}$$

be a chain of strong reductions of $\tau_\alpha$. Let $\tau_{r_1}$ and $\tau_{r_2}$ be any two topologies in $C_\varphi$ such that, say $\tau_{r_1}$ is strictly weaker than $\tau_{r_2}$ by one set. That is, $\tau_{r_1}$ is a strong reduction of $\tau_{r_2}$. Let

$$\tau_1 = \{f_\alpha^{-1}(G_{1i}) : G_{1i} \in \tau_{r_1}\}$$

and

$$\tau_2 = \{f_\alpha^{-1}(G2i) : G2i \in \tau_{r_2}\}.$$

Then clearly

$$\bigcap_{i=1}^{n} f_\alpha^{-1}(G_{1i}) = f_\alpha^{-1}\left[\bigcap_{i=1}^{n}(G_{1i})\right] \in \tau_1$$

as $\tau_{r_1}$ is closed under finite intersections. That is, $\tau_1$ is closed under finite intersections. Also

$$\cup f_\alpha^{-1}(G_{1i}) = f_\alpha^{-1}(\cup G_{1i}) \in \tau_1,$$

implying that $\tau_1$ is closed under arbitrary unions. It is easy to see that $\emptyset, X \in \tau_1$ as $\emptyset, X_\alpha \in \tau_{r_1}$. Hence $\tau_1$ is a topology on $X$, corresponding to $\tau_{r_1}$. Similarly $\tau_2$ is a topology on $X$ corresponding to $\tau_{r_2}$. It is easy to see that both $\tau_1$ and $\tau_2$ are weaker than $\tau$.

It is obvious that $\tau_1$ is weaker than $\tau_2$ and (by Lemma 1.1) that $\tau_1$ is only one set less than $\tau_2$. That is, $\tau_1$ is a strong reduction of $\tau_2$.

As $\tau_{r1}$ and $\tau_{r2}$ in $C_\varphi$ are arbitrary it follows that there corresponds to $C_\varphi$ a chain $C$ of topologies on $X$ of pair-wise comparable topologies which can be arranged in such a way that each one is strictly weaker than the next by only one set. If we let the elements of $C$ to represent the (hypothetical) range space $(X_\alpha, \tau_\alpha)$—one after the other—in the collection of sub-base for weak topologies on $X$ while leaving the other range spaces unchanged, the required chain of weaker weak topologies on $X$ will emerge.

■

## II. Summary & Conclusions

1. The concepts of strong, normal and weak reduction of topologies are introduced.
2. We proved that the discrete topology of a set X cannot be reduced in the strong sense of the cardinality of X is greater than 2.
3. We proved that the usual topology of the set of real numbers cannot be reduced in the strong sense.
4. The concepts of base reduction and strong base reduction of topologies are introduced.
5. The concepts of strong and weak extensions of topologies are introduced.
6. Strong base extension, weak base extension and base extension of topologies are introduced.
7. We established the conditions which guarantee that a topology is extensible, weakly extensible, base or weakly base extensible.
8. The idea of a chain of reductions for a topology is introduced, as well as the idea of comparable and equivalent chains of reductions.
9. We obtained the conditions for a weak topology to have a chain of reductions.
10. Ample examples are given at appropriate places to illustrate the ideas discussed.

Theorem 1.2 indicates that a fixed family of functions can generate a family of pairwise comparable weak topologies. Further research may now embark on finding more considerations for this result. This is part of the developments in our published works titled *Comparison Theorems for Weak Topologies* (see references below).

**Note**

So far, all the chains of strong reduction of topologies given in this paper are countable. The question then arises as to whether there can be an uncountable chain of strong reductions of some topology. For example, can an uncountable chain of strong reductions be obtained for the usual topology of $R$? Further, if a range topology for a weak topology has an uncountable chain of strong reductions, what is the implication of this on the weak topology? That is, does the weak topology in this case inherit this property? Can we characterize the weak topologies for which there exist families of other weak topologies which are chains of strong reductions of the given weak topologies? Answers to these questions are as yet unknown and open a window for further research in this area.

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