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A Novel Approach to Nonlinear Volterra-Fredholm Integral Equations Using Abaoub Shkheam Decomposition Method

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Abstract: In this study, we introduce a novel approach to the solution of a nonlinear Volterra -Fredholm integral equations by applying the Adomian decomposition method under the effect of the Abaoub- Shkheam transform. We demonstrate the existence and uniqueness of the solution in Banach space and illustrate this idea with an example.

Keywords: Abaoub Shkheam-Transform, Volterra- Integral Equations, Fredholm, Integral Equations

I. Introduction

We examine a Volterra Fredholm integral equations of second kind that is nonlinear, given by

$$u(x) = f(x) + \lambda \int_a^x k_1(x,t) F_1\big(u(t)\big)dt + \lambda \int_a^b k_2(x,t) F_2\big(u(t)\big)dt, \tag{1}$$

where $u(x)$ is the unknown function that will be determined, $k_i(x,t), i = 1,2,$ and the function $f(x)$ are given real-valued functions. The functions $F_i(u)$ are given nonlinear functions of u, λ is the parameter, a and b are constants.

In 2020, Abaoub and Shkheam introduced a novel integral transform known as the Abaoub Shkheam transform [14], which they utilized to address linear Volterra integral equations [15]. The following year, Shkheam and collaborators used the Abaoub Shkheam trans form to solve the Linear Volterra Integro-Differential Equation of the First Kind [16]. In 2022, Asmaa Mubayrash applied this transform to solve partial differential equations [19], and in 2022 Asma Mubayrash and her colleagues using this transform for Solving Partial Differential Equations [17]. Building on this work, in 2023, Suad Zali employed the transform to tackle linear partial integro-differential equations [20], while Nagah Elbhilil and colleagues used it to solve Volterra Integral and Volterra Integro-Differential Equations [18].

The Adomian Decomposition Method (ADM) is a brand-new, extremely powerful method that Adomian [2],[21] first presented in the early 1980s for solving a wide range of equations, including integral, differential, partial differential, and linear and non-linear algebraic equations [22]-[31]. The solution series has shown to rapidly converge using this strategy. The non-linear term is broken down into a set of specialized polynomials known as Adomian's polynomials in order for it to function. Our main goal in this paper is to solve non-linear Volterra Fredholm integral equations by using the Combined Aboub Shkheam Transform-Adomian Decomposition Method.

The Abaoub Shkheam Decomposition Method

The nonlinear Volterra-Fredholm integral equation with difference kernels is expressed as follows:

$$u(x) = f(x) + \lambda \int_a^x k_1(x-t) F_1\big(u(t)\big)dt + \lambda \int_a^b k_2(x-t) F_2\big(u(t)\big)dt, \tag{2}$$

Abaoub Shkheam transform method can solves the nonlinear Volterra-Fredholm integro differential equation (2). We utilize the Abaoub Shkheam transform on bothe sides of (2) we get

$$Q[u(x)] = Q[f(x)] + \lambda Q\left[\int_a^x k_1(x-t) F_1\big(u(t)\big)dt\right] + \lambda Q\left[\int_a^b k_2(x-t) F_2\big(u(t)\big)dt\right] \tag{3}$$

using convolution theorem of the Q-Transform in (3), we obtain

$$Q[u(x)] = Q[f(x)] + \lambda v Q[k_1(x)] Q\big[F_1\big(u(x)\big)\big] + \lambda v Q[k_2(x)] Q\big[F_2\big(u(x)\big)\big] \tag{4}$$

The Adomian decomposition method, along with Adomian polynomials, can be employed to tackle equation (4) and address the nonlinear term $F_i\,\big(u(t)\big), i = 1,2$. Initially, we represent the linear term $u(x)$ on the left side as an infinite series of components, given by:

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$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{5}$$

where the components $u_n(x),\ n \geq 0$ will be determined recursively. However, the nonlinear terms $F_1\big(u(t)\big)$ and $F_2\big(u(t)\big)$ on the right side of equation (4) will be expressed as infinite series involving Adomian polynomials $A_n(t)$ and $B_n(t)$, respectively, in the following form:

$$F_1(u) = \sum_{n=0}^{\infty} A_n(t)\ , \qquad F_2(u) = \sum_{n=0}^{\infty} B_n(t), \tag{6}$$

where A_n and B_n, $n \geq 0$ are defined by

$$A_n(t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[F_1\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]\right]_{\lambda=0} \quad , \quad B_n(t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[F_2\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]\right]_{\lambda=0} .$$

where the Adomian polynomials A_n can be computed for various forms of nonlinearity. Specifically, for a given nonlinear function $F_1\big(u(t)\big)$, the Adomian polynomials are defined as follows:

$$A_0 = F_1(u_0)$$

$$A_1 = u_1 {F_1}'(u_0)$$

$$A_2 = u_2 {F_1}'(u_0) + \frac{1}{2!}\ u_1^2 {F_1}''(u_0)$$

$$A_3 = u_3 {F_1}'(u_0) + u_1 u_2 {F_1}''(u_0) + \frac{1}{3!}\ u_1^3 {F_1}'''(u_0)$$

$$A_4 = u_4 {F_1}'(u_0) + \left(\frac{1}{2!} {u_2}^2 + u_1 u_3\right) {F_1}''(u_0)\ + \frac{1}{2!}\ u_1^2 u_2 {F_1}'''(u_0) + \frac{1}{4!}\ u_1^4 {F_1}^{(iv)}(u_0).$$

Similarly, We can evaluated the Adomian polynomials B_n of the nonlinear function $F_2\big(u(x)\big)$ as following

$$B_0 = F_2(u_0)$$

$$B_1 = u_1 {F_2}'(u_0)$$

$$B_2 = u_2 {F_2}'(u_0) + \frac{1}{2!}\ u_1^2 {F_2}''(u_0)$$

$$B_3 = u_3 {F_2}'(u_0) + u_1 u_2 {F_2}''(u_0) + \frac{1}{3!}\ u_1^3 {F_2}'''(u_0)$$

$$B_4 = u_4 {F_2}'(u_0) + \left(\frac{1}{2!}\ u_2^2 + u_1 u_3\right) {F_2}''(u_0) + \frac{1}{2!}\ u_1^2 u_2 {F_2}'''(u_0) + \frac{1}{4!}\ u_1^4 {F_2}^{(iv)}(u_0).$$

Substituting equation (5) and equation (6) into equation (4) leads to:

$$Q\left[\sum_{n=0}^{\infty} u_n(x)\right] = Q[f(x)] + \lambda v Q[k_1(x)] Q\left[\sum_{n=0}^{\infty} A_n(t)\right] + \lambda v Q[k_2(x)] Q\left[\sum_{n=0}^{\infty} B_n(t)\right].$$

The recursive relation is presented by using the Adomian decomposition method

$$\begin{cases} Q[u_0(x)] = Q[f(x)] \\ Q[u_{n+1}(x)] = \lambda v Q[k_1(x)] Q\left[\sum_{n=0}^{\infty} A_n(t)\right] + \lambda v Q[k_2(x)] Q\left[\sum_{n=0}^{\infty} B_n(t)\right] \end{cases} \tag{7}$$

When the first part of equation (7) is applied with the inverse Abaoub-Shkheam transform, $u_0(x)$ is obtained. Determine $u_0(x)$ and $u_1(x)$ yields $A_1(x)$and $B_1(x)$ is used to evaluate $u_2(x)$ and so on. This leads to the complete determination of the components of $u_n(x)$,n ≥ 0 upon applying the second part of Eq. (7). The series solution follows immediately after applying Eq. (5). The obtained series solution may converge to an exact solution, if such a solution exists.

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Existence and Uniqueness Analysis of Solutions

We shall introduce and prove the results regarding the existence and uniqueness of solution to Equation (2). In order to prove these results, we first present the suitable hypotheses.

$(\boldsymbol{H_1})$ $F_j(u) \in C^\infty$in a neighborhood of u, and for any n (the derivatives of $F_j(u)$ at u are bounded in norm);

$$\left\|F_j^{(n)}(u)\right\| \leq M_j, \qquad j = 1,2.$$

$(\boldsymbol{H_2})$ There exists a constant $M < 1$ such that, for any u_i in Banach space $C([a,b] \|.\|)$.

$$\|u_i\| \quad \leq M, i = 1,2, \dots$$

$(\boldsymbol{H_3})$ We suppose that for all $a \leq t \leq x \leq b$ the kernels $k_j(x-t), j = 1,2$ are satisfies the conditions:

$$\left\{\int_a^x \left|k_j(x-t)\right|^2 dt\right\}^{\frac{1}{2}} < C_j.$$

$(\boldsymbol{H_4})$ The functions $F_j(u)$, satisfying the Lipschitz condition:

$$\left\|F_j(u) - F_j(v)\right\| \leq L_j\|u-v\|, \; L_j > 0, j = 1,2.$$

Theorem 3.1 [32]

Under the previous hypotheses $(\boldsymbol{H_1})$and $(\boldsymbol{H_2})$, the series $\sum_{n=0}^{\infty} A_n$ is absolutely convergent and, furthermore

$$\|A_n\| \leq \left(\exp\left(\pi\sqrt{\frac{2}{3}n}\right)\right) M' M^n ,$$

where M' is the minimal of M_j.

Theorem 3.2 (*Existence and Uniqueness*)

Suppose that $(\boldsymbol{H_1}), (\boldsymbol{H_2}),$ $(\boldsymbol{H_3}),$ and $(\boldsymbol{H_4})$ hold. If $0 < \alpha, \beta < 1$, where

$$\alpha = |\lambda|(C_1 M_1 + C_2 M_2),$$

and

$$\beta = |\lambda|(C_1 L_1 + C_2 L_2).$$

Then there exists a unique solution $u(x) \in C$ $([a,b], \|.\|)$ to Eq.(2).

Proof:

By using the Adomian decomposition method, we get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^x \left[k_1(x-t) \sum_{n=0}^{\infty} A_n(t)\right] dt + \lambda \int_a^b \left[k_2(x-t) \sum_{n=0}^{\infty} B_n(t)\right] dt.$$

$$\begin{cases} u_0(x) = f(x) \\ u_n(x) = \lambda \int_a^x k_1(x-t) A_{n-1}(t) dt + \lambda \int_a^b k_2(x-t) B_{n-1}(t) dt, n \geq 1 \end{cases} \qquad (8)$$

taking the norm of equation (8), yields

$$\|u_n(x)\| \leq |\lambda| \left\|\int_a^x k_1(x-t) A_{n-1}(t) dt\right\| + |\lambda| \left\|\int_a^b k_2(x-t) B_{n-1}(t) dt\right\|$$
$$\leq |\lambda| \left\{\int_a^x |k_1(x-t) A_{n-1}(t)|^2 dt\right\}^{\frac{1}{2}} + |\lambda| \left\{\int_a^b |k_2(x-t) B_{n-1}(t)|^2 dt\right\}^{\frac{1}{2}},$$

utilizing the Cauchy- Schwarz inequality, we get

$$\|u_n(x)\| \leq |\lambda| \left\{\int_a^x |k_1(x-t)|^2 dt\right\}^{\frac{1}{2}} \left\{\int_a^b |A_{n-1}(t)|^2 dt\right\}^{\frac{1}{2}}$$

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$$+|\lambda|\left\{\int_a^b |k_2(x-t)|^2 dt\right\}^{\frac{1}{2}}\left\{\int_a^b |B_{n-1}(t)|^2 dt\right\}^{\frac{1}{2}}$$

$$=|\lambda|\left\{\int_a^x |k_1(x-t)|^2 dt\right\}^{\frac{1}{2}}\|A_{n-1}(t)\|+|\lambda|\left\{\int_a^b |k_2(x-t)|^2 dt\right\}^{\frac{1}{2}}\|B_{n-1}(t)\|, n\geq 1\,.$$

Now by using hypotheses $(\boldsymbol{H_1})$, $(\boldsymbol{H_2})$, $(\boldsymbol{H_3})$ and the theorem (3.1), we get

$$\|u_n(x)\|\leq|\lambda|C_1\left(\exp\left(\pi\sqrt{\frac{2}{3}(n-1)}\right)\right)M_1M^{n-1}+|\lambda|C_2\left(exp\left(\pi\sqrt{\frac{2}{3}(n-1)}\right)\right)M_2M^{n-1}$$

$$=\alpha M^{n-1}\,exp\left(\pi\sqrt{\frac{2}{3}(n-1)}\right)$$

Since $0<\alpha<1$, then

$$|\lambda|<\frac{1}{C_1M_1+C_2M_2}. \tag{9}$$

Under the aforementioned condition (9), the bound ensures that the infinite series $\sum_{n=0}^{\infty} u_n(x)$converges uniformly. Consequently, the function $u(x)$can be expressed as:

$$u(x)=\sum_{n=0}^{\infty} u_n(x).$$

Since each $u_n(x)$ is continuous, $u(x)$ inherits this property, demonstrating that $u(x)$ is continuous and convergent. This confirms the existence of a solution to Equation (2).

Now, we shall prove the uniqueness solution, suppose that Eq. (2) has two solutions $u(x)$, and $v(x)$. Applying the norm on both sides of Eq.(2), we obtain

$$\|u(x)-v(x)\|\leq|\lambda|\left\|\int_a^x k_1(x-t)[F_1(u(t))-F_1(v(t))]dt\right\|+|\lambda|\left\|\int_a^b k_2(x-t)\big[F_2(u(t))-F_2(v(t))\big]dt\right\|$$

$$\leq|\lambda|\left\{\int_a^x |k_1(x-t)[F_1(u)-F_1(v)]|^2 dt\right\}^{\frac{1}{2}}\qquad+|\lambda|\left\{\int_a^b |k_2(x-t)[F_2(u)-F_2(v)]|^2\,dt\right\}^{\frac{1}{2}}.$$

By Cauchy-Schwarz inequality and Using hypotheses $(\boldsymbol{H_1})$, $(\boldsymbol{H_3})$ and $(\boldsymbol{H_4})$, we have

$$\|u(x)-v(x)\|\ \leq|\lambda|\left\{\int_a^x |k_1(x-t)|^2 dt\right\}^{\frac{1}{2}}\left\{\int_a^b |F_1(u)-F_1(v)|^2 dt\right\}^{\frac{1}{2}}$$

$$+|\lambda|\left\{\int_a^b |k_2(x-t)|^2 dt\right\}^{\frac{1}{2}}\left\{\int_a^b |F_2(u)-F_2(v)|^2 dt\right\}^{\frac{1}{2}}$$

$$\leq|\lambda|C_1\|F_1(u)-F_1(v)\|\ +|\lambda|C_2\|F_2(u)-F_2(v)\|.$$

We apply the Lipschitz condition

$$\|u(x)-v(x)\|\ \leq|\lambda|[C_1L_1+C_2L_2]\ \|u(x)-v(x)\|\ \leq\beta\|u(x)-v(x)\|,$$

hence

$$(\beta-1)\|u(x)-v(x)\|\geq 0.$$

Since $0<\beta<1$ this implies that $u(x)=v(x)$. ◻

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Theorem 3.3 (*Convergence*).

Assume that $(\boldsymbol{H_3})$, and $(\boldsymbol{H_4})$, then the series solution (5) of the equation (2) converges to the exact solution provided $\|u_1(x)\| < \infty$, and $0 < \beta < 1$.

Proof

Let $(C[a,b], \|.\|)$ denote the Banach space of all continuous real-valued functions defined on $[a,b]$. Consider the sequence of partial sums S_n defined by:

$$S_n = \sum_{i=0}^{n} u_i(x)$$

which represents the partial sums of the series solution (5).Since

$$F_1\left(\sum_{i=0}^{\infty} u_i(t)\right) = \sum_{i=0}^{\infty} A_i(t), \qquad F_2\left(\sum_{i=0}^{\infty} u_i(t)\right) = \sum_{i=0}^{\infty} B_i(t).$$

So,

$$F_1(S_n) = \sum_{i=0}^{n} A_i(t), \quad F_2(S_n) = \sum_{i=0}^{n} B_i(t). \tag{10}$$

Let S_n and S_m be arbitrary partial sums with $n \geq m$, then

$$\|S_n - S_m\| = \left\| \sum_{i=m+1}^{n} \left[\lambda \int_a^x k_1(x-t) A_{i-1}(t)\, dt + \lambda \int_a^b k_2(x-t) B_{i-1}(t) dt \right] \right\|$$

$$= \left\| \lambda \int_a^x k_1(x-t) \sum_{i=m}^{n-1} A_i(t)\, dt + \lambda \int_a^b k_2(x-t) \sum_{i=m}^{n-1} B_i(t)\, dt \right\|$$

from (10), we have

$$\|S_n - S_m\| \leq |\lambda| \left\{ \int_a^x |k_1(x-t)[F_1(S_{n-1}) - F_1(S_{m-1})]|^2 dt \right\}^{\frac{1}{2}}$$

$$+|\lambda| \left\{ \int_a^b |k_2(x-t)[F_2(S_{n-1}) - F_2(S_{m-1})]|^2 dt \right\}^{\frac{1}{2}}$$

$$\leq |\lambda| \left\{ \int_a^x |k_1(x-t)|^2 dt \right\}^{\frac{1}{2}} \left\{ \int_a^b |F_1(S_{n-1}) - F_1(S_{m-1})|^2 dt \right\}^{\frac{1}{2}}$$

$$+|\lambda| \left\{ \int_a^b |k_2(x-t)|^2 dt \right\}^{\frac{1}{2}} \left\{ \int_a^b |F_2(S_{n-1}) - F_2(S_{m-1})|^2 dt \right\}^{\frac{1}{2}}$$

$$\leq \big[|\lambda|[C_1 L_1 + C_2 L_2]\big] \|S_{n-1} - S_{m-1}\|$$

$$= \beta \|S_{n-1} - S_{m-1}\|. \tag{11}$$

Let $n = m + 1$, then

$$\|S_{m+1} - S_m\| \leq \beta \|S_m - S_{m-1}\| \leq \beta^2 \|S_{m-1} - S_{m-2}\| \leq \cdots \leq \beta^m \|S_1 - S_0\|$$

and since

$$\|S_n - S_m\| \leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \cdots + \|S_n - S_{n-1}\|$$

$$\leq \beta^m [1 + \beta + \beta^2 + \cdots + \beta^{n-m-1}] \|S_1 - S_0\| \leq \beta^m \left(\frac{1 - \beta^{n-m}}{1 - \beta} \right) \|u_1\|. \tag{12}$$

Since $0 < \beta < 1$, then

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$$\|S_n - S_m\| \leq \frac{\beta^m}{1-\beta}\|u_1\| . \tag{13}$$

But $\|u_1\| < \infty$, so, as $m \to \infty$, then

$$\|S_n - S_m\| \to 0 .$$

We conclude that $\{S_n\}$ is a Cauchy sequence in Banach space, the convergence of the sequence is equivalent to the convergence of the series. □

4. Applications:

An examination of the following example demonstrates the Abaoub Shkheam decomposition method for solving the nonlinear Volterra Fredholm equations.

Example. Consider the $\mathrm{NVFIE_s}$ the nonlinear Volterra-Fredholm integro differential equation of the second kind

$$u(x) = e^x - e^{2x} - x\,e^{3x} + \int_0^x e^{3(x-t)}u^3(t)\,dt + \int_0^1 e^{2(x-t)}u^2(t)\,dt, \tag{14}$$

Taking the Q-ttransform of both sides of the equation (14) gives

$$\begin{aligned} Q[u(x)] &= \frac{v}{1-u\,v} - \frac{v}{1-2\,u\,v} - \frac{uv^2}{(3uv-1)^2} + Q[e^{3x} * u^3(x)] + Q[e^{2x} * u^2(x)] \\ &= \frac{v}{1-u\,v} - \frac{v}{1-2\,u\,v} - \frac{uv^2}{(3uv-1)^2} + \frac{uv}{1-3u\,v}Q[u^3(x)] \\ &+ \frac{uv}{1-2\,u\,v}Q[u^2(x)]. \end{aligned} \tag{15}$$

Utilized the Adomian decomposition method to both sides of the equation (15), gives:

$$Q\left[\sum_{n=0}^{\infty} u_n(x)\right] = \frac{v}{1-u\,v} - \frac{v}{1-2\,u\,v} - \frac{uv^2}{(3uv-1)^2} + \frac{uv}{1-3uv}Q\left[\sum_{n=0}^{\infty} A_n(x)\right] + \frac{uv}{1-2\,uv}Q\left[\sum_{n=0}^{\infty} B_n(x)\right].$$

The Adomian decomposition method presents the recursive relation:

$$Q[u_0(x)] = \frac{v}{1-uv} - \frac{v}{1-2\,uv} - \frac{uv^2}{(3uv-1)^2}$$

$$Q\{u_{n+1}(x)\} = \frac{uv}{1-3uv}Q\left[\sum_{n=0}^{\infty} A_n(x)\right] + \frac{uv}{1-2\,uv}Q\left[\sum_{n=0}^{\infty} B_n(x)\right], n \geq 0, \tag{16}$$

where $A_n(x)$ and $B_n(x)$ are the Adomian polynomials for the nonlinear term $u^3(x)$, and $u^2(x)$, respectively. The Adomian polynomials for $F_1\big(u(x)\big) = u^3(x)$ and $F_2\big(u(x)\big) = u^2(x)$ are given by:

$$A_0 = F(u_0) = u_0^3,$$

$$A_1 = u_1 F'(u_0) = 3u_0^2 u_1$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) = 3u_0^2 u_2 + 3u_0 u_1^2,$$

$$\vdots$$

$$B_0 = F(u_0) = u_0^2,$$

$$B_1 = u_1 F'(u_0) = 2u_0 u_1,$$

$$B_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) = 2u_0 u_2 + u_1^2,$$

$$\vdots$$

Substituting in Eq.(16), we obtain

$$u_0 = e^x - e^{2x} - xe^{3x}$$

$$Q\{u_1(x)\} = \frac{uv}{1-3uv}Q[A_0(x)] + \frac{uv}{1-2\,uv}Q[B_0(x)],$$

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$$u_1(x) = Q^{-1}\left\{\frac{uv}{1-3uv}Q[\,u_0^3] + \frac{uv}{1-2\,uv}Q[u_0^2]\right\}$$

$$u_1(x) = Q^{-1}\left\{\frac{uv}{1-3uv}Q[\,(e^x - e^{2x} - \,xe^{3x})^3] + \frac{uv}{1-2\,uv}Q[(e^x - e^{2x} - \,xe^{3x})^2]\right\}$$

$$u_1(x) = xe^{3x}(1-3xe^{4x}) + \frac{e^{6x}}{3}\left(\frac{23}{9}x - 5x^2 - \frac{58}{81}\right) + \frac{e^{4x}}{3}\left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{9} - \frac{1}{9}\right).$$

that converges to the exact solution

$$u(x) = e^x.$$

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