INTERNATIONAL JOURNAL OF LATEST TECHNOLOGY IN ENGINEERING, MANAGEMENT & APPLIED SCIENCE (IJLTEMAS)
ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue X,October 2024

# Approximation in Weighted Space with Generalized Max-Product Type Favard-Szász-Mirakyan-Durrmeyer Operators

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**DOI : https://doi.org/10.51583/IJLTEMAS.2024.131025**

**Received: 02 November 2024; Accepted: 12 November 2024; Published: 18 November 2024**

**Abstract:** In this paper, we explore the uniform approximation of functions using Generalized Favard- Szász-Mirakyan - Durrmeyer operators of the max-product type with specific exponential weighted spaces. We analyze the approximation rate with an appropriate continuity modulus.

**Keywords:** Durrmeyer Operators**,** Exponential Weighted Spaces, Maximum product operators, Approximation rate.

## I. Introduction

Let $f$ be a function defined on $[0,\infty)$. The Favard- Szász-Mirakyan operators $S_n$ applied to $f$ are given by

$$S_n(f,x) = e^{-(nx)} \sum \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \qquad (1)$$

The approximation properties of Favard- Szász-Mirakyan operator have been studied by many authors; a few examples [2],[6],[13] and [14].

$x \in [0,\infty)$ ve $n \in \mathbb{N}$ The Durrmeyer-type modification, as defined by Mazhar and Totik [12], is given by,

$$S_n(f,x) = e^{-(nx)} \sum \frac{(nx)^k}{k!} f(\frac{k}{n})\, n \int_0^\infty P_{n,k}(t) f(t) dt,$$

where $P_{n,k}(t) = e^{-(nx)} \frac{(nx)^k}{k!}$ and $f \in C[0,\infty)$.

The linear structure is not always preserved or sufficient in approximation operators. Furthermore, standard algebraic operations such as addition and subtraction may be inadequate. To address these issues, maximum-product type approximation operators were introduced, utilizing the maximum algebraic operation. These operators are nonlinear and positive, thereby expanding the range of tools available in approximation theory. There is extensive research in this field [1-5],[8-11].

The max-product type Favard-Szász-Mirakyan operator is defined as follows:

$$\mathrm{F}_n^{(M)}(f,x) = \frac{\bigvee_{n=0}^{\infty} \frac{(nx)^k}{k!} f(\frac{k}{n})}{\bigvee_{n=0}^{\infty} \frac{(nx)^k}{k!}}, \qquad n \in \mathbb{N} \qquad (2)$$

where $x \in [0,\infty)$ and $f:[0,\infty) \to \mathbb{R}_+$ is considered bounded on $[0,\infty)$ (Bede et al. [4])

Many modifications of the max-product type Favard-Szász-Mirakyan operator have been studied. The Durrmeyer generalization of these operators has been investigated in [3] and [4] for continuous and bounded functions defined on the interval [0,1] and a different generalization has also been explored in [5].

In this paper, we demonstrate that the operators $F_n$ can be utilized for uniform approximation with the weight $\upsilon = e^{\alpha\phi(x)}$, where $\phi(x) = \sqrt{x}$. We also examine the rate of convergence of these operators. determine the rate of convergence of these operators to the identity operator.

Any sequence of positive linear operators can be used for the uniform approximation of functions across a wide range of weights defined by $\upsilon = e^{\alpha\phi(x)}$, which are associated with certain operators ( [7] and [8]).

Let $\phi: I \to J$ be a function defined over a non-compact interval $I \subseteq \mathbb{R}$. This function $\phi$ is continuous and strictly monotonic. The interval $J \subseteq \mathbb{R}$ corresponds precisely to $\phi(I)$. For $\alpha \geq 0$, we denote the space of continuous functions as follows:

$$C_{\alpha,\phi} = \{f \in C(I), \text{there exists } M > 0 \text{ such that } (\frac{|f(I)|}{e^{\alpha\phi(x)}}) \text{ for every } x \in I\}.$$

This space can be endowed with the norm

$$\|f\| = sup_{x\in I}\frac{|f(I)|}{e^{\alpha\phi(x)}}.$$

The modulus of continuity $\omega_{\alpha,\phi}(f;\cdot)$ is given for every $f \in C_{\alpha,\phi}$ and $\delta \geq 0$ as follows

$$\omega_{\alpha,\phi}(f;x) = sup_{\substack{x\in I\\ |\phi(t)-\phi(x)|\leq\delta}}\frac{|f(t)-f(t)|}{max(e^{\alpha\phi(t)},e^{\alpha\phi(x)})}$$

where the supremum is taken for all $x \in I$ and $t \in I$ such that

$$\phi(t) \in (\phi(t)-\delta,\phi(t)+\delta)\cap\phi(I).$$

For $\alpha = 0$ and $\phi(x) = x$, we get the usual modulus of continuity $\omega(f;\delta)$, (see, [8]).

**II. Auxiliary Results**

In this section, we define generalized Favard-Szász-Mirakyan-Durrmeyer operators and present some supplementary results to investigate the approximation properties of these operators.

**Definition 1** $f\ :\ [0,\infty) \to \mathbb{R}_+$, an integrable function and $x \in [0,\infty)$.

$$D_n^{(M)}(f,x) = \frac{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}\cdot n\int_0^\infty P_{n,k}(t)\,f\left(\frac{k}{n}\right)dt}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}\cdot n\int_0^\infty P_{n,k}(t)\,dt}, n \in N$$

the operator is referred to as max-product type Favard-Szász-Mirakyan -Durrmeyer operator, where $P_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}$ ,(see [11]).

**Definition 2** $f$: $[0,\infty) \to \mathbb{R}_+$, an integrable function and $x \in [0,\infty)$,

$$N_n^M\left(e^{\alpha\sqrt{\frac{k}{b_n}}};x\right) = \frac{\bigvee_{n=0}^{\infty}\frac{(b_nx)^k}{k!}n\int_0^\infty P_{n,k}(t)\,e^{\alpha\sqrt{\frac{k}{b_n}}}dt}{\bigvee_{n=0}^{\infty}\frac{(b_nx)^k}{k!}n\int_0^\infty P_{n,k}(t)\,dt}, n \in \mathbb{N}$$

the operator defined by the above equality is referred to as the generalized max-product type Favard-Szász-Mirakyan-Durrmeyer Operators**,** where $P_{n,k}(t) = e^{-(nx)}\frac{(nx)^k}{k!}$.

**Lemma 1** For $\alpha \geq 0$ and $(b_n)$ being a positive, increasing, and unbounded real sequence, we consider the following intervals for $n \in \mathbb{N}$.

$I_0 = [0, e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)})$ and $I_k = [ke^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{k}-\sqrt{k-1})}, (k+1)e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{k+1}-\sqrt{k})})$, $k \geq 1$.

The intervals are non-empty, disjoint, and collectively they span the positive half line.

Indeed $I_k = (k+1)e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{k+1}-\sqrt{k})} - ke^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{k}-\sqrt{k-1})} \geq 0.$

**Lemma 2** If $\forall b_nx \in I_j$ then,

$\bigvee_{n=0}^{\infty}\frac{(b_nx)^k}{k!}n\int_0^\infty e^{-(nt)}\frac{(nt)^k}{k!}e^{\alpha\sqrt{\frac{k}{b_n}}}dt = \frac{(b_nx)^j}{j!}n\int_0^\infty e^{-(nt)}\frac{(nt)^j}{j!}e^{\alpha\sqrt{\frac{j}{b_n}}}dt.$

**Proof** Let us denote $a_k = \bigvee_{n=0}^{\infty}\frac{(b_nx)^k}{k!}n\int_0^\infty P_{n,k}(t)\,e^{\alpha\sqrt{\frac{k}{b_n}}}dt.$

We get

$$0 \leq a_{k+1} \leq a_k \Leftrightarrow nx \in \left[0,(k+1)e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{k+1}-\sqrt{k})}\right)$$

$\Rightarrow k = 0,$

$$0 \leq a_1 \leq a_0$$

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue X,October 2024

$$0 \le b_n x \int_0^\infty n t e^{-(nt)} \, e^{\alpha\sqrt{\frac{1}{b_n}}} dt \le \int_0^\infty e^{-(nt)} \, dt$$

$$0 \le b_n x \Gamma(2) e^{\alpha\sqrt{\frac{1}{b_n}}} \le \Gamma(1)$$

$$0 \le b_n x \le e^{-\alpha\sqrt{\frac{1}{b_n}}}$$

$$b_n \in \left[0, e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)}\right)$$

$k = 3,$

$$0 \le a_4 \le a_3$$

$$0 \le \frac{(b_n x)^4}{4!} \int_0^\infty \frac{(nt)^4}{4!} e^{-nt} e^{\alpha\sqrt{\frac{4}{b_n}}} dt \le \frac{(b_n x)^3}{3!} \int_0^\infty \frac{(nt)^3}{3!} e^{-nt} e^{\alpha\sqrt{\frac{3}{b_n}}} dt$$

$$0 \le \frac{b_n x}{4} \frac{e^{\alpha\sqrt{\frac{4}{b_n}}}}{4} \int_0^\infty (nt)^4 e^{-nt} dt \le e^{\alpha\sqrt{\frac{3}{b_n}}} \int_0^\infty (nt)^3 e^{-nt} dt$$

$$0 \le \frac{b_n x}{4} \frac{e^{\alpha\sqrt{\frac{4}{b_n}}}}{4} \Gamma(5) \le e^{\alpha\sqrt{\frac{3}{b_n}}} \Gamma(4)$$

$$0 \le b_n x \le 4 e^{-\frac{\alpha}{\sqrt{b_n}}(\sqrt{4}-\sqrt{3})}$$

$$\vdots$$

$$b_n x \in \left[0, (k+1) e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{k+1}-\sqrt{k})}\right),$$

as opposed to $b_n x \in I^0 \Rightarrow$

$$b_n x \le e^{-\frac{\alpha}{\sqrt{b_n}}}$$

$$b_n x n e^{\frac{\alpha}{\sqrt{b_n}}} \Gamma(2) \le n \Gamma(2) = n \Gamma(1)$$

$$b_n x n e^{\frac{\alpha}{\sqrt{b_n}}} \int_0^\infty u e^{-u} du \le n \int_0^\infty e^{-u} du$$

$$b_n x n e^{\frac{\alpha}{\sqrt{b_n}}} \int_0^\infty t e^{-nt} dt \le \int_0^\infty e^{-nt} dt$$

$$a_1 \le a_0$$

$b_n x \in I_k \Rightarrow$

$$b_n x \le (k+1) e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{(k+1)}-\sqrt{k})}$$

$$\frac{b_n x}{k+1} e^{\alpha\frac{\sqrt{k+1}}{\sqrt{b_n}}} \le e^{\alpha\frac{\sqrt{k}}{\sqrt{b_n}}}$$

$$\frac{(b_n x)^{k+1}}{(k+1)!} n e^{\alpha\frac{\sqrt{k+1}}{\sqrt{b_n}}} \Gamma(k+2) \le \frac{(b_n x)^k}{k!} n e^{\alpha\frac{\sqrt{k}}{\sqrt{b_n}}} \Gamma(k+1)$$

$$\frac{(b_n x)^{k+1}}{(k+1)!} n \int_0^\infty \frac{(nt)^{k+1}}{(k+1)!} e^{-nt} e^{\alpha\frac{\sqrt{k+1}}{\sqrt{b_n}}} dt \le \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-nt} e^{\alpha\frac{\sqrt{k}}{\sqrt{b_n}}} dt$$

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue X,October 2024

$$a_{k+1} \leq a_k.$$

**Lemma 3** For $\alpha \geq 0$ and $(b_n)$ being a positive, increasing, and unbounded real sequence, $n \in \mathbb{N}$ we have,

$$N_n^M\left(e^{\alpha\sqrt{\frac{k}{b_n}}}; x\right) \leq e^{\alpha\sqrt{x}} e^{\frac{\alpha^2}{b_n}}.$$

**Proof** For $\forall b_n x \in I_j$ and from lemma 2,

$$N_n^M\left(e^{\alpha\sqrt{\frac{k}{b_n}}}; x\right) = \frac{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} e^{\alpha\sqrt{\frac{k}{b_n}}} dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} dt}$$

$$= \frac{\frac{(b_n x)^j}{j!} n \int_0^\infty e^{-(nt)} \frac{(nt)^j}{j!} e^{\alpha\sqrt{\frac{j}{b_n}}} dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} dt}$$

Let $m = \lfloor b_n x \rfloor$, from lemma 2, for $\alpha = 0$ we have

$$\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} = \bigvee_{n=0}^{\infty} \frac{(b_n x)^m}{m!}$$

and

$$\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} dt = \bigvee_{n=0}^{\infty} \frac{(b_n x)^m}{m!} n \int_0^\infty \frac{(nt)^m}{m!} e^{-(nt)} dt$$

$$e^{-\alpha\sqrt{x}} N_n^M\left(e^{\alpha\sqrt{\frac{k}{b_n}}}; x\right) = \frac{\frac{(b_n x)^j}{j!} n \int_0^\infty \frac{(nt)^j}{j!} e^{-(nt)} e^{\alpha\sqrt{\frac{j}{b_n}}} dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^m}{m!} n \int_0^\infty \frac{(nt)^m}{m!} e^{-(nt)} dt}$$

$$= \frac{m!\,(b_n x)^j}{j!\,(b_n x)^m} e^{\alpha\left(\sqrt{\frac{j}{b_n}} - \sqrt{x}\right)} \frac{\Gamma(j+1) m!}{j!\,\Gamma(m+1)}$$

$$= \frac{m!}{j!} (b_n x)^{j-m} e^{\frac{\alpha}{\sqrt{b_n}}\left(\sqrt{j} - \sqrt{(b_n x)}\right)}$$

$$= \frac{m!}{j!} (b_n x)^{j-m} e^{\frac{\alpha}{\sqrt{b_n}\left(\sqrt{j} + \sqrt{(b_n x)}\right)}(j - b_n x)}$$

$$b_n x \in \left[ j e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\sqrt{j} - \sqrt{j-1}\right)}, (j+1) e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\sqrt{j+1} - \sqrt{j}\right)} \right)$$

and

$$m \leq b_n x \leq (j+1) e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\sqrt{j+1} - \sqrt{j}\right)} \leq (j+1),$$

we have $m \leq (j+1)$.

Using the inequality $1 - e^{-x} \leq x$ we get ,

$$j - b_n x \leq j - j e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\sqrt{j} - \sqrt{j-1}\right)} \leq j \left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\sqrt{j} - \sqrt{j-1}\right)$$

$$= j \left(\frac{\alpha}{\sqrt{b_n}}\right) \frac{j - j + 1}{\sqrt{j} + \sqrt{j-1}} \leq \alpha \sqrt{\frac{j}{b_n}}$$

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$$e^{-\alpha\sqrt{x}}N_n^M\left(e^{\alpha\sqrt{\frac{k}{b_n}}};x\right)=\left(\frac{b_nx}{m+1}\right)^{j-m}e^{\frac{\alpha}{\sqrt{b_n}\left(\sqrt{j}+\sqrt{(b_nx)}\right)}(j-b_nx)}$$

$$\leq\left(\frac{b_nx}{m+1}\right)^{j-m}e^{\frac{\alpha^2}{b_n(\sqrt{j}+\sqrt{(j-1)})}}\leq e^{\frac{\alpha^2}{b_n}}.$$

**Remark 1** We have $N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}},e^{\alpha\sqrt{x}};x\right)\right)\leq e^{\alpha\sqrt{x}}e^{\frac{\alpha^2}{b_n}}$ for $\forall x\geq 0$ and $\alpha>b_n$. Indeed

$$N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}},e^{\alpha\sqrt{x}};x\right)\right)=max\left(N_n^M\left(e^{\alpha\sqrt{\frac{k}{b_n}}};x\right),N_n^M\left(e^{\alpha\sqrt{x}};x\right)\right)$$

from Lemma 3 and $N_n^M\left(e^{\alpha\sqrt{\frac{k}{b_n}}};x\right)\leq e^{\alpha\sqrt{x}}e^{\frac{\alpha^2}{b_n}}$,

$$N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}},e^{\alpha\sqrt{x}};x\right)\right)=max\left(e^{\alpha\sqrt{x}}e^{\frac{\alpha^2}{b_n}},e^{\alpha\sqrt{x}}\right)=e^{\alpha\sqrt{x}}e^{\frac{\alpha^2}{b_n}}$$

**Remark 2** For $\forall\Psi(x)=\sqrt{x}$, for every $\forall f\in C_{\Psi,\alpha}$ for this reason $N_n^M\in C_{\Psi,\alpha}$,

$$\begin{aligned}|N_n^M(f;x)|&\leq N_n^M(|f|;x)\\&\leq N_n^M\left(\|f\|e^{\alpha\sqrt{t}};x\right)\\&=\|f\|N_n^M\left(e^{\alpha\sqrt{t}};x\right)\\&\leq\|f\|e^{\alpha\sqrt{x}}e^{\frac{\alpha^2}{b_n}}.\end{aligned}$$

**Lemma 4** For all $x\geq 0$ and $n\in\mathbb{N}$ , the following inequality holds:

$$\frac{\bigvee_{k\leq b_nx}^{\infty}\frac{(b_nx)^k}{k!}n\int_0^{\infty}\frac{(nt)^k}{k!}e^{-(nt)}\left(\sqrt{b_nx}-\sqrt{k}\right)dt}{\bigvee_{n=0}^{\infty}\frac{(b_nx)^k}{k!}n\int_0^{\infty}\frac{(nt)^k}{k!}e^{-(nt)}dt}\leq 1.$$

**Proof** $m=\lfloor b_nx\rfloor$ and

$$\bigvee_{n=0}^{\infty}\frac{(b_nx)^k}{k!}n\int_0^{\infty}\frac{(nt)^k}{k!}e^{-(nt)}\,dt=\bigvee_{n=0}^{\infty}\frac{(b_nx)^m}{m!}n\int_0^{\infty}\frac{(nt)^m}{m!}e^{-(nt)}\,dt.$$

If $m=0$ then $0=\lfloor b_nx\rfloor<1$ and $k\leq b_nx$ because it is $k=0$,

$$\frac{\bigvee_{k\leq b_nx}^{\infty}n\int_0^{\infty}e^{-(nt)}\left(\sqrt{b_nx}-0\right)dt}{n\int_0^{\infty}e^{-(nt)}\,dt}=\max\sqrt{b_nx}\leq 1.$$

If $m=1$ then $1=\lfloor b_nx\rfloor<2$ and $k\leq b_nx$ because it is $k=0,1$,

$$\frac{\bigvee_{k\leq b_nx}^{\infty}(b_nx)n\int_0^{\infty}nte^{-(nt)}\left(\sqrt{b_nx}-1\right)dt}{\bigvee_{n=0}^{\infty}(b_nx)\,n\int_0^{\infty}nte^{-(nt)}\,dt}=\max\left(\sqrt{b_nx}-1\right)\leq 1$$

If $m\geq 2$ then $r_k=\frac{(b_nx)^k}{k!}n\int_0^{\infty}\frac{(nt)^k}{k!}e^{-(nt)}\left(\sqrt{b_nx}-\sqrt{k}\right)$ and if $k=0$

$$\frac{r_0}{\bigvee_{n=0}^{\infty}(b_nx)\,n\int_0^{\infty}nte^{-(nt)}\,dt}=\frac{\sqrt{b_nx}}{\frac{(b_nx)^m}{m!}n\int_0^{\infty}\frac{(nt)^m}{m!}e^{-(nt)}\,dt}$$

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$$= \frac{m!\sqrt{b_n x}}{(b_n x)^m} = \frac{1}{\sqrt{b_n x}} \frac{2}{b_n x} \cdots \frac{m}{b_n x} \leq 1.$$

It remains to evaluate the maximum of $r_k$, for $k \geq 1$

$$r_k = \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} \left(\sqrt{b_n x} - \sqrt{k}\right)$$

$$= \frac{(b_n x)^k}{k!} \frac{b_n x - k}{\sqrt{b_n x} + \sqrt{k}} \leq \frac{(b_n x)^k}{k!} \frac{b_n x - k}{\sqrt{b_n x} + 1}.$$

Let us denote $c_k = \frac{(b_n x)^k}{k!} \frac{b_n x - k}{\sqrt{b_n x} + 1}$ , we get

$$\frac{c_k}{c_{k-1}} = \frac{b_n x}{k} \frac{b_n x - k}{\sqrt{b_n x} + 1} \leq 1 \Leftrightarrow b_n x (b_n x - k) \leq k\left(\sqrt{b_n x} + 1\right)$$

$(b_n x - k)^2 \leq k \Rightarrow b_n x \leq k + \sqrt{k}$ for every integer $k \leq b_n x$ we get $r_k \leq r$ by taking

$k = 2,3\ldots, c_2 \leq c_1 \Leftrightarrow b_n x \in [2, 2 + \sqrt{2})$, and so on. Let us denote

$$j_k = \left[k + \sqrt{k}, k + 1 + \sqrt{k+1}\,\right), \forall k = 1,2,3 \ldots$$

We deduce that if $b_n x \in j_k$ then $c_k \leq c_{k-1} 1\}$ for $\forall k \geq 1$ we obtain

$$\frac{\bigvee_{1 \leq k \leq b_n x}^{\infty} r_k}{\frac{(b_n x)^m}{m!} n \int_0^\infty \frac{(nt)^m}{m!} e^{-(nt)}\, dt} \leq \frac{\bigvee_{1 \leq k \leq b_n x}^{\infty} c_k}{\frac{(b_n x)^m}{m!}} \leq \frac{\frac{(b_n x)^j}{j!}}{\frac{(b_n x)^m}{m!}} \frac{b_n x - k}{\sqrt{b_n x} + 1}$$

$$\leq \frac{b_n x - k}{\sqrt{b_n x} + 1} \leq \frac{j + 1 + \sqrt{j+1} - j}{\sqrt{j + \sqrt{j} + 1}}$$

$$\leq \frac{1 + \sqrt{j+1}}{\sqrt{j + \sqrt{j} + 1}} \leq 1.$$

**Lemma 5** For $\forall x \geq 0$ and $\alpha \geq 0$, n∈ℕ, there exists a $b_n \geq \alpha^2$ such that the following inequality holds:

$$\frac{\bigvee_{k > b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} e^{\alpha \sqrt{\frac{k}{b_n}}} \left(\sqrt{k} - \sqrt{b_n x}\right) dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)}\, dt} \leq e^{\alpha\sqrt{x}} e^{\frac{2\alpha}{\sqrt{b_n}}}$$

**Proof** We have,

$$\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)}\, dt = \bigvee_{n=0}^{\infty} \frac{(b_n x)^m}{m!} n \int_0^\infty \frac{(nt)^m}{m!} e^{-(nt)}\, dt$$

and $m = \lfloor b_n x \rfloor$ for $m = 0$,

$$e^{-\alpha\sqrt{x}} \frac{\bigvee_{k > b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} e^{\alpha \sqrt{\frac{k}{b_n}}} \left(\sqrt{k} - \sqrt{b_n x}\right) dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)}\, dt}$$

$$\leq \frac{\bigvee_{k \geq 1}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} e^{\alpha \sqrt{\frac{k}{b_n}} - \alpha\sqrt{x}} \left(\sqrt{k}\right) dt}{n \int_0^\infty e^{-(nt)}\, dt}$$

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$k = 1,$

$$\leq \frac{\bigvee_{k\geq 1}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^{\infty} \frac{(nt)^k}{k!} e^{-(nt)} e^{\frac{\alpha}{\sqrt{b_n}}(\sqrt{k}-\sqrt{b_n x})} (\sqrt{k}) dt}{n \int_0^{\infty} e^{-(nt)} dt}$$

$$\leq \frac{e^{\frac{\alpha\sqrt{k}}{\sqrt{b_n}}}\sqrt{k}}{k!}$$

$$\leq e^{\frac{2\alpha}{\sqrt{b_n}}}.$$

For $m \geq 1$ and $k \geq 2$ we have $d_k = \frac{e^{\frac{\alpha\sqrt{k}}{\sqrt{b_n}}}\sqrt{k}}{k!} \leq max(d_1, d_2) <,$

$$e^{-\alpha\sqrt{x}} \frac{\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^{\infty} \frac{(nt)^k}{k!} e^{-(nt)} e^{\alpha\sqrt{\frac{k}{b_n}}} (\sqrt{k} - \sqrt{b_n x}) dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^{\infty} \frac{(nt)^k}{k!} e^{-(nt)} dt}$$

$$= \frac{\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^{\infty} \frac{(nt)^k}{k!} e^{-(nt)} e^{\frac{\alpha}{\sqrt{b_n}}(\sqrt{k}-\sqrt{b_n x})} \frac{k - b_n x}{\sqrt{k} + \sqrt{b_n x}} dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^{\infty} \frac{(nt)^k}{k!} e^{-(nt)} dt}$$

$$\leq \frac{\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^{\infty} \frac{(nt)^k}{k!} e^{-(nt)} e^{(\frac{k-b_n x}{\sqrt{k}+\sqrt{b_n x}})} \frac{k - b_n x}{\sqrt{k} + \sqrt{b_n x}} dt}{\frac{(b_n x)^m}{m!} n \int_0^{\infty} \frac{(nt)^m}{m!} e^{-(nt)} dt}$$

$$\frac{k - b_n x}{\sqrt{k} + \sqrt{b_n x}} \leq \frac{k - b_n x}{2\sqrt{b_n x}}$$

$$\leq \frac{\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^{\infty} \frac{(nt)^k}{k!} e^{-(nt)} e^{\left(\frac{k-b_n x}{2\sqrt{b_n x}}\right)} \frac{k - b_n x}{2\sqrt{b_n x}} dt}{\frac{(b_n x)^m}{m!} n \int_0^{\infty} \frac{(nt)^m}{m!} e^{-(nt)} dt}$$

$$\leq \frac{\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} e^{(\frac{k-b_n x}{2\sqrt{b_n x}})} \frac{k-b_n x}{2\sqrt{b_n x}}}{\frac{(b_n x)^m}{m!}},$$

$$\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} e^{(\frac{k-b_n x}{2\sqrt{b_n x}})} \frac{k-b_n x}{2\sqrt{b_n x}} = \frac{(b_n x)^j}{j!} e^{(\frac{j-b_n x}{2\sqrt{b_n x}})} \frac{j-b_n x}{2\sqrt{b_n x}}$$

and for $b_n x \in \left[ j e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{j}-\sqrt{j-1})}, (j+1) e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{j+1}-\sqrt{j})} \right)$ we have

$$\frac{\frac{(b_n x)^j}{j!} e^{(\frac{j-b_n x}{2\sqrt{b_n x}})} \frac{j - b_n x}{2\sqrt{b_n x}}}{\frac{(b_n x)^m}{m!}} \leq \left(\frac{b_n x}{m+1}\right)^{j-m} e^{\frac{\alpha}{\sqrt{b_n}}\left(\frac{\frac{\alpha\sqrt{j}}{\sqrt{b_n}}}{2\sqrt{j e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{j}-\sqrt{j-1})}}}\right)} \left(\frac{\frac{\alpha\sqrt{j}}{\sqrt{b_n}}}{2\sqrt{j e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{j}-\sqrt{j-1})}}}\right)$$

$$j - b_n x = j - j e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)(\sqrt{j}-\sqrt{j-1})}$$

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue X,October 2024

$$= j\left(1 - e^{-\left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\sqrt{j}-\sqrt{j-1}\right)}\right) \le j\left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\sqrt{j} - \sqrt{j-1}\right)$$

$$= j\left(\frac{\alpha}{\sqrt{b_n}}\right)\left(\frac{j - j + 1}{\sqrt{j} + \sqrt{j-1}}\right) \le \frac{\alpha\sqrt{j}}{\sqrt{b_n}}$$

$$\le e^{\frac{\alpha}{\sqrt{b_n}}\left(\frac{\alpha\sqrt{e^{\frac{\alpha}{\sqrt{b_n}}\left(\frac{1}{(\sqrt{j}+\sqrt{j-1})}\right)}}}{2\sqrt{b_n}}\right)\left(\frac{\alpha\sqrt{e^{\frac{\alpha}{\sqrt{b_n}}\left(\frac{1}{(\sqrt{j}+\sqrt{j-1})}\right)}}}{2\sqrt{b_n}}\right)}$$

$$\le e^{\frac{2\alpha}{\sqrt{b_n}}}.$$

**Lemma 6** For $\forall x, \alpha \ge 0$, $n \in \mathbb{N}$ and $b_n \ge \alpha^2$ we have

$$N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}}, e^{\alpha\sqrt{x}}\right)\left|\sqrt{\frac{k}{b_n}} - \sqrt{x}\right|; x\right) \le \frac{1}{\sqrt{b_n}} e^{\alpha\sqrt{x}} e^{\frac{2\alpha}{\sqrt{b_n}}}.$$

**Proof** We have

$$N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}}, e^{\alpha\sqrt{x}}\right)\left|\sqrt{\frac{k}{b_n}} - \sqrt{x}\right|; x\right) = max(A_n, B_n)$$

where

$$A_n = \frac{\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} e^{\alpha\sqrt{\frac{k}{b_n}}} \left(\sqrt{\frac{k}{b_n}} - \sqrt{x}\right) dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)}\, dt}$$

$$= \frac{1}{\sqrt{b_n}} \frac{\bigvee_{k>b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} e^{\alpha\sqrt{\frac{k}{b_n}}} \left(\sqrt{k} - \sqrt{b_n x}\right) dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)}\, dt}$$

$$\le \frac{1}{\sqrt{b_n}} e^{\alpha\sqrt{x}} e^{\frac{2\alpha}{\sqrt{b_n}}}$$

from lemma 5,

$$B_n = \frac{\bigvee_{k\le b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} \left(\sqrt{x} - \sqrt{\frac{k}{b_n}}\right) dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)}\, dt}$$

$$= \frac{1}{\sqrt{b_n}} \frac{\bigvee_{k\le b_n x}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)} \left(\sqrt{b_n x} - \sqrt{k}\right) dt}{\bigvee_{n=0}^{\infty} \frac{(b_n x)^k}{k!} n \int_0^\infty \frac{(nt)^k}{k!} e^{-(nt)}\, dt}$$

$$\le \frac{1}{\sqrt{b_n}} e^{\alpha\sqrt{x}}.$$

**Approximation in Weighted Space with Generalized Max-Product Type Favard-Szász-Mirakyan-Durrmeyer Operators**

**Theorem 1** For $\Psi(x) = \sqrt{x}$ and $\forall f \in C_{\Psi,\alpha}$ ,

$$\|N_n^M(f;x)-f\|_{\Psi,\alpha} \le \left(e^{\frac{\alpha^2}{b_n}}+e^{\frac{2\alpha}{\sqrt{b_n}}}\right)\omega_{\Psi,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right), \forall n\in\mathbb{N}, b_n\ge\alpha^2.$$

**Proof** $N_n^M(1;x)=1$ we get,

$$|N_n^M(f;x)-f| \le |N_n^M(f(t);x)-f(x)N_n^M(1;x)|$$

$$= |N_n^M(f(t);x)-N_n^M(f(x);x)|$$

$$\le \left|N_n^M\big(f(t)-f(x)\big);x)\right|$$

$$\le N_n^M(|f(t)-f(x)|;x)$$

and $|f(t)-f(x)| \le \omega_{\Psi,\alpha}(f,\delta_n) \le (\frac{1+|\sqrt{t}-\sqrt{x}|}{\delta_n})\omega_{\Psi,\alpha}(f,\delta_n)$ according to this

$$\le N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}},e^{\alpha\sqrt{x}}\right)\left(1+\frac{\left|\sqrt{\frac{k}{b_n}}-\sqrt{x}\right|}{\delta_n}\right);x\right)\omega_{\Psi,\alpha}(f,\delta_n)$$

$$\le N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}},e^{\alpha\sqrt{x}}\right);x\right)\omega_{\Psi,\alpha}(f,\delta_n)+\frac{1}{\delta_n}N_n^M\left(max\left(e^{\alpha\sqrt{\frac{k}{b_n}}},e^{\alpha\sqrt{x}}\right)\left(\left|\sqrt{\frac{k}{b_n}}-\sqrt{x}\right|\right);x\right)\omega_{\Psi,\alpha}(f,\delta_n)$$

using remark 1 and lemma 6 also taking the value $\delta_n=(\frac{1}{\sqrt{b_n}})$ we obtain

$$\le\left(e^{\alpha\sqrt{x}}e^{\frac{\alpha^2}{b_n}}+\frac{\sqrt{b_n}}{\sqrt{b_n}}e^{\alpha\sqrt{x}}e^{\frac{2\alpha}{\sqrt{b_n}}}\right)\omega_{\Psi,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)$$

$$=\left(e^{\frac{\alpha^2}{b_n}}+e^{\frac{2\alpha}{\sqrt{b_n}}}\right)e^{\alpha\sqrt{x}}\omega_{\Psi,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)$$

$$e^{-\alpha\sqrt{x}}|N_n^M(f;x)-f|=\left(e^{\frac{\alpha^2}{b_n}}+e^{\frac{2\alpha}{\sqrt{b_n}}}\right)\omega_{\Psi,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)$$

which proves the theorem.

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