INTERNATIONAL JOURNAL OF LATEST TECHNOLOGY IN ENGINEERING, MANAGEMENT & APPLIED SCIENCE (IJLTEMAS)

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue V, May 2024

On the Exponential Diophantine Equation $2^x + 1,245^y = z^2$

Theeradach Kaewong, Wariam Chuayjan, and Sutthiwat Thongnak*

Department of Mathematics and Statistics, Faculty of Science and Digital Innovation, Thaksin University, Phatthalung 93210, Thailand.

***Corresponding Author**

DOI : https://doi.org/10.51583/IJLTEMAS.2024.130516

Received: 18 May 2024; Accepted: 31 May 2024; Published: 15 June 2024

Abstract: Let $x, y$and z be non-negative integers. We solve the exponential Diophantine equation $2^x + 1,245^y = z^2$. The result indicates that the equation has a unique solution,$(x, y, z) = (3,0,3)$.

Keywords: divisibility; exponential Diophantine equation; modular arithmetic; Divisibility; Catalan's conjecture; quadratic residue; Legendre symbol properties:

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I. Introduction

Let a and $b$be positive integers. The exponential Diophantine equation $a^x + b^y = z^2$, where x, y and $z$are unknown non-negative integers, was solved by many researchers. The examples can be seen in [1, 5, 7, 9-14]. In 2023, S. Aggarwal et al. solved the two exponential Diophantine equations, including $143^x + 85^y = z^2$and $143^x + 485^y = z^2$. The proof was based on the modular arithmetic method and Catalan's conjecture. Another equation $255^x + 323^y = z^2$ was proposed (see [2 - 3, 15]). Recently, the exponential Diophantine equation $147^x + 741^y = z^2$ has been proposed (see [16]). They showed that the equation has no solution. After that, S. Aggarwal et al. showed that the exponential Diophantine equation $10^x + 400^y = z^2$ has no solution (see [4]). Then T. Kaewong et al. studied $305^x + 503^y = z^2$. They proved that the equation has no solution (see [8]). In this work, we solve the exponential Diophantine equation $2^x + 1,245^y = z^2$ where x, y and z are non-negative integers**.**

II. Preliminaries

In this section, we introduce basic knowledge applied in this proof.

Definition 2.1 [6] Let p be an odd prime and $gcd(a, p) = 1$. If the quadratic congruence $x^2 \equiv a (mod\, p)$has a solution, then a is said to be a quadratic residue of p**.** Otherwise, a is called a quadratic nonresidue of p**.**

Definition 2.2 [6] Let p be an odd prime and let $gcd(a, p) = 1$. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if} a \text{is} a \text{quadratic residue of} p \\ -1 & \text{if} a \text{is} a \text{quadratic nonresidue of} p \end{cases}$$

Theorem 2.3 (Catalan**'**s conjecture [10]) Let a, b, x, and$y$be integers**.** The Diophantine equation $a^x - b^y = z^2$ with $min\{a, b, x, y\} > 1$has the unique solution $(a, b, x, y) = (3,2,2,3)$.

Theorem 2.4. [6] Let p be an odd prime and let a and b be integers that are relatively prime to p. Then the Legendre symbol has the following properties:

(a) If$a \equiv b (mod\, p)$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(b) $\left(\frac{a^2}{p}\right) = 1$.

(c) $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} (mod\, p)$.

(d) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

(e) $\left(\frac{1}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$.

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue V, May 2024

III. Result

Theorem 3.1 The exponential Diophantine equation $2^x + 1{,}245^y = z^2$ where x, y and $z$are non-negative integers has a unique solution $(x, y, z) = (3,0,3)$.

Proof: Let x, y and $z$be non-negative integers such that

$$2^x + 1{,}245^y = z^2. \quad (1)$$

We separate into four cases as follows.

Case 1: $x = y = 0$. (1) becomes $z^2 = 2$, which is impossible.

Case 2: $x = 0$and$y > 0$. From (1), we obtain $z^2 = 1 + 1{,}245^y$. Then it implies that $z^2 \equiv 2 (mod\, 4)$. This is a contradiction because $z^2 \equiv 0{,}1 (mod\, 4)$.

Case 3: $x > 0$and$y = 0$. We have

$$z^2 - 2^x = 1. \quad (2)$$

If $x = 1$, then (2) becomes $z^2 = 3$, impossible.

If $x > 1$, then (2) implies $z > 1$. From Catalan's conjecture, it follows that $(x, z) = (3,3)$. Thus, the solution is $(x, y, z) = (3,0,3)$.

Case 4: $x > 0$and$y > 0$. From (1), we obtain $z^2 \equiv (-1)^x (mod\, 3)$. Because $z^2 \equiv 0{,}1 (mod\, 3)$, we have $(-1)^x \equiv 0{,}1 (mod\, 3)$. Then $x$is an even positive integer, yielding $x = 2$ or $x \geq 4$. If $x = 2$, then (1) becomes $z^2 = 4 + 1{,}245^y$, implying $z^2 \equiv 5 (mod\, 3\,11)$. Then, 5 is a quadratic residue of 311. It yields $\left(\frac{5}{311}\right) = 1$but $\left(\frac{5}{311}\right) = -\left(\frac{311}{5}\right) = -\left(\frac{1}{5}\right) = -1$. Thus, $\left(\frac{5}{311}\right) = 1$, yields $1 = -1$, which is a contradiction. If $x \geq 4$, then we have $z^2 \equiv 5^y (mod\, 8)$. We can see that $5^y \equiv 5 (mod\, 8)$ if y is an odd positive integer. Then, we have .$z^2 \equiv 5 (mod\, 8)$. It is impossible because $z^2 \equiv 0{,}1{,}4 (mod\, 8)$. Thus, y must be an even positive integer. Let $y = 2k$, $\exists k \in \mathbb{Z}^+$. From (1), we have$2^x = z^2 - 1{,}245^{2k}$ or $2^x = (z - 1{,}245^k)(z + 1{,}245^k)$. There exists $\alpha \in \{0,1,2, \dots, x\}$ such that $z - 1{,}245^k = 2^\alpha$ and $z + 1{,}245^k = 2^{x-\alpha}$ where $x - \alpha > \alpha$. It follows that

$$2 \cdot 1{,}245^k = 2^{x-\alpha} - 2^\alpha. \quad (3)$$

Then (3) implies that $\alpha = 0$ or $\alpha = 1$. In the case of$\alpha = 0$, (3) becomes $2 \cdot 1{,}245^k = 2^x - 1$, which is impossible. In the case of$\alpha = 1$, we can write (3) as $1{,}245^k = 2^{x-2} - 1$ or

$$2^{x-2} - 1{,}245^k = 1. \quad (4)$$

If $k = 1$, then we have $2^{x-2} = 1{,}246$, impossible. If $k > 1$, then (4) implies that $x > 3$. By Theorem 2.3, it follows that **(4)** has no solution. From all cases, $(x, y, z) = (3,0,3)$is a unique non-negative integer solution to the equation. □

IV. Conclusion

We have solved the exponential Diophantine equation $2^x + 1{,}245^y = z^2$ where x, y and z are non-negative integers. The knowledge in Number theory including Catalan's conjecture, modular arithmetic, divisibility, quadratic residue and Legendre symbol properties has been applied in the proof, we have found that the equation has a unique solution, $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\mathbf{3}, \mathbf{0}, \mathbf{3})$.

Acknowledgment

We would like to thank the reviewers for their careful reading of our manuscript and their useful comments.

References

1. Acu, D., (2007) On a Diophantine Equation, General Mathematics, 15(4), 145-148.
2. Aggarwal, S., Swarup, C., Gupta, D., and Kumar, S., (2023) Solution of the Diophantine Equation $143^x + 85^y = z^2$, International Journal of Progressive Research in Science and Engineering, 4(02), 5 – 7.
3. Aggarwal, S., Kumar, S., Gupta, D., and Kumar, S., (2023) Solution of the Diophantine Equation $143^x + 485^y = z^2$, International Research Journal of Modernization in Engineering Technology and Science, 5(02), 555 – 558.
4. Aggarwal, S., Pandey, R., and Kumar, S., (2024) Solution of the Exponential Diophantine Equation $10^x + 400^y = z^2$, International Journal of Latest Technology in Engineering, Management & Applied Science (IJLTEMAS), 8(2), 38 – 40.
5. Burshtein, N., (2019) On Solution to the Diophantine Equations $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$ with Positive Integers x, y, z, Annals of Pure and Applied Mathematics, 19(1), 75-77.
6. Burton, D. M., (2011) Elementary Number Theory, Seventh Edition, The McGraw-Hill companies.

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue V, May 2024

7. Jeyakrishnan, G. and Komahan, G., (2017) More on the Diophantine Equation $27^x + 2^y = z^2$, International Journal for Scientific Research & Development, 4(2), 166-167.
8. Kaewong, T., Thongnak, S.and Chuayjan, W., (2024) On the Exponential Diophantine Equation $305^x + 503^y = z^2$, International Journal of Latest Technology in Engineering, Management & Applied Science (IJLTEMAS), 8(2), 79 – 81.
9. Kumar, S. and Aggarwal, S., (2021) On the Exponential Diophantine Equation $439^p + 457^q = r^2$, Journal of Emerging Technologies and innovative Research (JETIR), 8(3), 2357 –2361.
10. Mihailescu, P., (2004) Primary Cyclotomic Units and a Proof of Catalan' s Conjecture, Journal fur die Reine und Angewandte
11. Mathematik, 572, 167–195.
12. Pakapongpun, A.and Chattae, B., (2022) On the Diophantine equation $p^x + 7^y = z^2$, where $p$is Primes and x, y, z are non-negative integers, International Journal of Mathematics and Computer Science, 17(4), 1535-1540.
13. Sroysang, B., (2014) More on the Diophantine Equation $3^x + 85^y = z^2$, International Journal of Pure and Applied Mathematics, 91(1), 131-134.
14. Suvarnamani, A., (2011) On two Diophantine Equations $4^x + 7^y = z^2$and$4^x + 11^y = z^2$, Science and Technology RMUTT Journal, 1(1), 25-28.
15. Tadee, S., (2022) On the Diophantine equation $p^x + (p + 14)^y = z^2$where $p, p + 14$are Primes, Annals of Pure and Applied Mathematics, 26(2), 125-130.
16. Viriyapong, N. and Viriyapong, C., (2023) On the Diophantine equation $255^x + 323^y = z^2$, International Journal of Mathematics and Computer Science, 18(3), 521 – 523.
17. Viriyapong, N. and Viriyapong, C., (2024) On the Diophantine equation $147^x + 741^y = z^2$, International Journal of Mathematics and Computer Science, 19(2), 445 – 447.