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On the Exponential Diophantine Equation $15^x - 17^y = z^2$

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Abstract: In this work, the exponential Diophantine equation $15^x - 17^y = z^2$, where x, y and z are non-negative integers, was studied and presented with the theorems governing its expressions. The result indicated that $(x, y, z) = (0,0,0)$ was a unique solution to the equation.

Keywords: divisibility; exponential Diophantine equation; modular arithmetic; prime number

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I. Introduction

Almost a decade ago, the exponential Diophantine equations of the form $a^x - b^y = z^2$, where a and b are positive integers, have been studied by researchers. They solved the individual equations based on knowledge of number theory. In 2018, Rabago [3] studied the exponential Diophantine equations $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$. He also proved all solutions to the equation $4^x - p^y = z^2$, where prime $p = z^q - 1$ or $p \equiv 3 \pmod 4$. The Mihailescus Theorem (Catalan's conjecture) was applied in these proofs. In 2019, S. Thongnak et al. [7] studied the equation $2^x - 3^y = z^2$. The result was obtained by using Mihailescus Theorem and modular arithmetic. In 2020, M. Buosi et al. [1] discovered non-negative integer solutions to $p^x - 2^y = z^2$ where prime $p = k^2 + 2$. In [2] A. Elshahed and H. Kamarulhaili proved all non-negative integer solutions to the equation $(4^n)^x - p^y = z^2$ are $(x, y, z, p) \in \{(k, 1, 2^{nk} - 1, 2^{nk+1} - 1)\} \cup \{(0,0,0,p)\}$. The exponential Diophantine equation $7^x - 5^y = z^2$ was solved by S. thongnak et al (see in [8]). They proved that equation has only the trivial solution $(x, y, z) = (0,0,0)$. Recently, many exponential Diophantine equations have been solved (see in [4], [5], [9-11]).

According to previous works, the exponential Diophantine equation problem is a challenging because there is no general method to determine solutions. In this work, we solve the exponential Diophantine equation $15^x - 17^y = z^2$, where x, y and z are non-negative integers. We applied the greatest common divisor and order of modular in the proof all solutions to the equation.

II. Preliminaries

In this section, we introduce basic knowledge applied in this proof.

Definition 2.1 [6] Let a and b be given integers, with at least one of them different from zero. The *greatest common divisor* of a and b, denoted by $\gcd(a,b)$, is the positive integer d satisfying the following:

(a) $d \mid a$ and $d \mid b$.

(b) If $c \mid a$ and $c \mid b$, then $c \le d$.

Definition 2.2 [6] If n is a positive integer and $\gcd(a,n) = 1$, the least positive integer k where $a^k \equiv 1 \pmod n$ is the order of a modulo n denoted by $\operatorname{ord}_n a$.

Lemma 2.1 [6] Let n be a positive integer, and a be an integer such that $\gcd(a,n) = 1$. If $\operatorname{ord}_n a = k$ and i, j are positive integers then $a^i \equiv a^j \pmod n$ if and only if $i \equiv j \pmod k$, $\exists i, j \in □^{+}$.

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Lemma 2.2 Let x be an integer. Then $x^2 \equiv 0,1 \pmod 3$.

Proof: Let x be an integer. There exist integers q and r such that $x = 11q + r,\ r = 0,1,2$. Thus, we have $x \equiv 0,1,2 \pmod 3$. It followed that $x^2 \equiv 0,1,4 \pmod 3$ or $x^2 \equiv 0,1 \pmod 3$. ☐

III. Result

Theorem 3.1. Let x, y and z be non-negative integers. The exponential Diophantine equation $15^x - 17^y = z^2$ has a unique solution: $(x, y, z) = (0,0,0)$.

Proof: Let x, y and z be non-negative integers such that

$$15^x - 17^y = z^2. \qquad (1)$$

For convenience, let us consider four cases as follows.

Case 1: $x = 0$ and $y = 0$. From (1), we have $z^2 = 0$ or $z = 0$, so the solution is $(0,0,0)$.

Case 2: $x = 0$ and $y > 0$. (1) becomes $1 - 7^y = z^2$, impossible, because of $1 - 7^y < 0$.

Case 3: $x > 0$ and $y = 0$. From (1), we have $z^2 = 15^x - 1$, which implies that $z^2 \equiv 2 \pmod 3$, which contradicts to Lemma 2.1.

Case 4: $x > 0$ and $y > 0$, (1) implies that $z^2 \equiv (-1)^x - 1 \pmod 4$. Since $z^2 \not\equiv -2 \pmod 4$, x is an even positive integer. Let $x = 2k,\ \exists k \in \mathbb{Z}^{+}$. By (1), we have $17^y = z^2 - 15^{2k}$ or $17^y = (z - 15^k)(z + 15^k)$. There exists $\alpha \in \{0,1,2,\ldots,y\}$, such that $z - 15^k = 17^\alpha$ and $z + 15^k = 17^{y-\alpha}$ where $\alpha < y - \alpha$. We obtain $2 \cdot 15^k = 17^\alpha + 17^{y-\alpha}$ or

$$2 \cdot 3^k \cdot 5^k = 17^\alpha \left(1 + 17^{y-2\alpha}\right). \qquad (2)$$

Because of $17 \nmid 2 \cdot 3^k \cdot 5^k$ and (2), we have $\alpha = 0$. It follows that

$$2 \cdot 3^k \cdot 5^k = 1 + 17^y.$$

(3)

(3) implies that $1 + (-1)^y \equiv 0 \pmod 3$, so y is an odd positive integer. By (3) again, we obtain that $2^y \equiv -1 \pmod 5$, then $2^y \equiv 2^2 \pmod 5$. Due to $\operatorname{ord}_5 2 = 4$, we applied Lemma 2.1. Then, we obtain $y \equiv 2 \pmod 4$. There exists $l \in \mathbb{Z}$ such that $y = 2 + 4l = 2(1 + 2l)$, which implies that $2 \mid y$. This is a contradiction because y is an odd positive integer.

☐

IV. CONCLUSION

We have solved the exponential Diophantine equation $15^x - 17^y = z^2$, where x, y and z are non-negative integers. This work was done using knowledge of Number Theory, including the greatest common divisor and order of modulo. The result shows that the equation has a unique solution (trivial solution), $(x, y, z) = (0,0,0)$.

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