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On the Exponential Diophantine Equation $2^x - 99^y = z^2$

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Abstract: In this work, we determine all non-negative integers solution (x, y, z) to the exponential Diophantine equation $2^x - 99^y = z^2$. The mathematical process bases on several theorems in Number theory, including the modular arithmetic, Divisibility and the Division Algorithm. The solution set to the equation is $(x, y, z) \in \{(0,0,0), (1,0,1)\}$.

Keywords: divisibility; the exponential Diophantine equation; modular arithmetic; Divisibility; the Division Algorithm:

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I. Introduction

The Diophantine equation is a classical problem in Number theory. For almost a decade, the famous general equation is $a^x - b^y = z^2$ where a and $b$are positive integers withx, y, and z are unknown non-negative integers. In 2018, the two equations $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$, were studied (see [2]). In 2019, the equation $\mathbf{2^x - 3^y = z^2}$ was proposed. They showed that the equation has three solutions,$(x, y, z) \in \{(0,0,0), (1,0,1), (2,1,1)\}$ (see [5]). In 2020, M. Buosi et al. studied $p^x - 2^y = z^2$ where prime $p = k^2 + 2$ (see [1]). Many equations were studied from 2021 to 2022 (see [4, 6-8]). Recently, S. Tadee studied the two equation, including$9^x - 3^y = z^2$ and$13^x - 7^y = z^2$. He proved that $(x, y, z) \in \{(r, 2r, 0)\}$ is the set of solutions to the $9^x - 3^y = z^2$, and $13^x - 7^y = z^2$ has only a trivial solution, $(0,0,0)$ (see [3]).

Previous works have shown that there is no general method for obtaining all solutions to the exponential Diophantine equations. They must prove their work based on mathematical processes and knowledge. This challenges mathematical researchers to study the individual equations. In this paper, we determined and proved all solutions the the exponential Diophantine equation $2^x - 99^y = z^2$ where x, y, and $z$are non-negative integers. The proof based on principles of mathematic and all solutions were given.

II. Preliminaries

In this section, we introduce basic knowledge applied in this proof.

Lemma 2.1 If x is an integer, then $x^2 \equiv 0,1 (mod\, 4)$.

Proof: Let x be an integer. We separate $x$into even and odd.

Case 1: $x$is even. We have$x = 2k$, where $k$is an integer. Then, it follows that $x^2 = 4k^2$ yielding $x^2 \equiv 0 (mod\, 4)$.

Case 2: $x$is odd. We have $x = 2k + 1$, where $k$is an integer. Then, it follows that $x^2 = 4(k^2 + k) + 1$ yielding $x^2 \equiv 1 (mod\, 4)$.

From two cases, we conclude that $x^2 \equiv 0,1 (mod\, 4)$.

Lemma 2.2 If x is a positive integer, then $3^x \equiv 1,3 (mod\, 8)$.

Proof: Let x be a positive integer. By the Division Algorithm, there exist non-negative integers q and r such that $x = 2q + r$ with $0 \le r < 2$. We obtain

$$\begin{aligned} 3^x &\equiv 3^{2q+r} \\ &\equiv 3^r (mod\, 8) \\ &\equiv 3^0, 3^1, 3^2 (mod\, 8) \\ &\equiv 1,3 (mod\, 8). \end{aligned}$$

Thus, we have $3^x \equiv 1,3 (mod\, 8)$ □

Lemma 2.3 If x is an integer, then $x^2 \equiv 0,1,4 (mod\, 8)$.

Proof: Let x be an integer. By the Division Algorithm, there exist integers q and r such that $x = 4q + r$ with $0 \le r < 4$. We obtain $x^2 = (2q^2 + qr)8 + r^2$ implying that $x^2 \equiv r^2 (mod\, 8)$. We can write

$$x^2 \equiv 0^2, 1^2, 2^2, 3^2 (mod\, 8)$$

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$\equiv 0,1,4 (mod\, 8)$.

Thus, we have $x^2 \equiv 0,1,4 (mod\, 8)$. □

III. Result

Theorem 3.1 The exponential Diophantine equation $2^x - 99^y = z^2$ where x, y and $z$are non-negative integers has two solutions, $(x, y, z) \in \{(0,0,0), (1,0,1)\}$.

Proof: Let x, y, and $z$be non-negative integers such that

$$2^x - 99^y = z^2 \quad (1)$$

we separate into four cases as follows.

Case 1: $x = y = 0$. By (1), we obtain $z = 0$. The one solution to the equation is $(0,0,0)$.

Case 2: $x = 0$and$y > 0$. From (1), we have $z^2 = 1 - 99^y < 0$, impossible.

Case 3: $x > 0$and$y = 0$. From (1), we have

$$z^2 = 2^x - 1. \quad (2)$$

From (2), if $x = 1$, then we have $z^2 = 1$. Thus $(1,0,1)$ is a solution to the equation. If $x \geq 2$, then we have $z^2 \equiv 3 (mod\, 4)$. This is impossible because $z^2 \equiv 0,1 (mod\, 4)$.

Case 4: $x > 0$and$y > 0$. Since $z^2 \geq 0$, (1) implies that $2^x \geq 99$, thus $x \geq 7$. From (1), we can write as $z^2 \equiv -3^y (mod\, 8)$. Since $z^2 \equiv 0,1,4 (mod\, 8)$, thus we have $-3^y \equiv 0,1,4 (mod\, 8)$ or $3^y \equiv 0,4,7 (mod\, 8)$. This is impossible because of $3^y \equiv 1,3 (mod\, 8)$. Therefore, the proof is complete. □

IV. Conclusion

In this work, we have solved the exponential Diophantine equation $2^x - 99^y = z^2$ where x, y, and z are non-negative integers. We derived three Lemmas for the proof and applied the modular arithmetic, the Divisibility, and the Division Algorithm. Finally, we have shown that $(x, y, z) \in \{(0,0,0), (1,0,1)\}$ are the solutions to the equation.

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