Equilibrium Solution of Two – Dimensional Non-Homogeneous Equations in the Theory of Elastic Mixtures

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DOI: https://doi.org/10.51583/IJLTEMAS.2024.130803

Received: 30 July 2024; Accepted: 10 August 2024; Published: 27 August 2024

Abstract: The problem of plane elasticity for a doubly connected body with inner and outer boundaries in a regular polygonal form with common centre and parallel sides has been studied. The sides of the polygon were exposed to external forces. The nature of the force term was determined by application of complex variable theory. Kolosov's method of solution was applied to obtain the biharmonic equation of the forcing term. The forces on the particle were studied under 2-dimensions from which the compatibility and equilibrium equations were derived. The compatibility and equilibrium equations were used to derive the force – stress relations. The results shows that there is a significant relationship between the angle of the force term on the plane of the particle and the stress state of the particle, which is in conformity with existing experimental results.

Keywords: Elasticity, equilibrium, forcing term, biharmonic, compatibility, stress state.

I. Introduction

The theory of elasticity describes deformable materials such as rubber, cloth, paper and flexible metals. It is often used to model the behavior of non-rigid curves, surfaces and solids as a function of time. Elasticity deformable models are active and respond naturally to applied forces, constraints, ambient, media and impenetrable obstacles, Terzopoulos *et al,* (1987).

One of the most efficient and elegant techniques of solving problems in the linear theory of elasticity is the method of complex stress functions which is mainly associated with Kolosov, (1909), Muskhelishvili (1966), Bock, and Gurlebeck, (2009). In particular, application of complex variable theory in solving elasticity problems resulted from the complex potential, which is peculiar to analytic complex functions. It is exceedingly fruitful for effective solution of boundary value problems and general functions that relates theoretically with Cauchy's integral formula and conformal mappings, Kapanadze and Gulna (2016). Chou and Pagano (2001) opined that one of the major problems in the theory of elasticity is that of determining the full strength of surfaces which aid in controlling the stress concentration both on the surface and at the boundaries of surfaces.

Recently, construction and engineering practices suffer major setbacks resulting from negligence, poor analysis and examination of materials and the stress strength of surfaces and contours on which the load/stress are imposed. Odishelidze and Kriado (2006) further established that investigation of stresses concentration near the contour of surfaces is one of the major problems in plane elasticity theory, especially in plate with a hole where the tangential-normal stresses and the tangential-normal moments can reach such values that cause destruction of plates or formation of plastic zones near the hole at some points. In cases of infinite domains, the minimum of maximum values of tangential-normal stresses will be obtained on such holes, where these values maintain constant (full strength holes).

A mixed problem of plane elasticity theory for doubly-connected domain with partially unknown boundary conditions was solved in Odishelidze *et al* (2015). The problem of plane elasticity theory for a doubly connected domain with partially unknown boundary was solved in Odishelidze (2015) using the methods of the theory of analytic complex functions. Boundary valued equation for force term in non-homogeneous equation of statics in the theory of elastic mixtures was solved in Udoh and Ndiwari, (2018) using Kolosov-Muskhelishvili formula for a displacement vector in an elastic mixture of homogeneous body. Biharmonic solution for a force term in a non-homogeneous equation of statics in the theory of elastic mixtures was provided by Ndiwari and Ongodiebi. (2020) using complex variable theory, where constant introduction of the force term at a fixed point on the plane directly affected the stability of the particle.

In this work, we considered a problem of plane elasticity for a doubly connected domain with inner and outer boundaries in a regular polygonal form with common center and parallel sides. The sides of the body were exposed to external unknown force and the boundary conditions were determined at equilibrium in order to ascertain the impact of the forcing term and its relationship with the stability of the isotropic elastic material. We derived the forcing term from the non-homogeneous equation of statics in the theory of elastic mixture. The unknown forces were analyzed in two-dimensions from stress function to derive the equilibrium, compatibility and biharmonic equations. The basic equation of elasticity was obtained using the compatibility equation and the stress-strain relation. The boundary equation of the unknown forcing term was derived and graphs generated to illustrate and explain the relationship between the angle of the forcing term and the stress state of the isotropic elastic material.

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue VIII, August 2024

Mathematical Formulation

We considered a homogeneous isotropic elastic body in a doubly connected domain D on the complex plane z = x + iy. Its outer and inner boundaries are L_o and L_1 respectively and form a rectangle with a common center z = 0 and parallel sides. The neighbourhood of the vertices of the inner rectangle are equal smooth arcs which are symmetric angles of equidistance from the centre as in Figure 1. We assumed that the edges of the isotropic elastic body are exposed to the external force in the form of load. We further assumed that both boundaries, L_0 of the elastic body and that of the hole are smooth and free from frictional forces. Under these assumptions, the normal displacements of the outer and inner boundaries are constant respectively, while the unknown arcs are exposed to external force. Our aim was to determine the equilibrium solution for the force term, F and the relationship between the angle of the force term on the plane of the particle and the stress state of the particle.

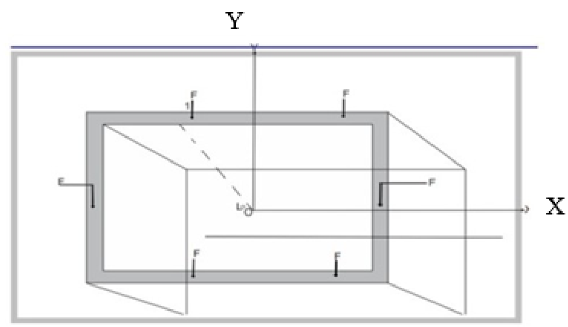

Figure 1. Isotropic elastic body

II. Method of Solution

To determine the force term F, we applied the non-homogeneous equation in the theory of elastic mixtures as our governing equation and adopt [Kolosov, (1909) and Muskhelishvili. (1966)]. The displacement components of the vector are represented in this theory by means of four arbitrary analytic functions as in [Ndiwari and Ongodiebi (2020)]. The basic non-homogeneous equations governing the theory of elastic mixture in 2-dimensions [Kapanadze and Gulna (2016)] is given by

$$a_1\Delta u' + b_1 graddivu' + c\Delta u'' + dgraddivu'' = \rho_1 F' = \Psi'$$

$$c\Delta u' + dgraddivu' + a_2\Delta u'' + b_2 graddivu'' = \rho_2 F'' = \Psi'' \qquad 1$$

Where Δ is the 2-dimensional Laplacian, grad and div are the principal operators of the field theory, ρ_1 and ρ_2 are the partial densities (positive constants of the mixture), F' and F'' are the mass forces, respectively; $u' = (u_1', u_2') = w'$ and $u'' = (u_1'', u_2'') = w''$ are the displacement vectors, Ψ' and Ψ'' denote the product of the partial density ρ and the mass force F, respectively, a_1,a_2, b_1, b_2, c and dare combination of constitutive constants characterizing the physical properties of the mixtures specified as

$$\left.\begin{array}{l} a_1 = \mu_1 - \lambda_5, \qquad a_2 = \mu_2 - \lambda_5, \\ b = \mu_2 + \lambda_2 + \lambda_5 + p^{-1}\alpha_2\rho_1, \qquad b_1 = \mu_1 + \lambda_1 + \lambda_5 + p^{-1}\alpha_2\rho_2, \\ c = \mu_3 + \lambda 5,\ d = \mu_3 + \lambda_4 - \lambda_5 - p^{-1}\alpha_2\rho_1,\ p = \rho_1 + \rho_2, \\ a_2 = \mu_2 - \lambda_2 + \lambda_5 + p^{-1}a_2\rho_1 \\ \alpha_2 = \lambda_3 - \lambda_4 \end{array}\right\} \qquad (2)$$

where μ_i, (i = 1,2,3) is the mixture's permeability constant and λ_i, (i =1,2,3,4,5) is the mixture's thermal conductivity constant.

Applying Complex Variables Theory

Applying complex variable theory, we solve Equation (1) as follows:

$$z = x_1 + ix_2 \qquad (3)$$

And its conjugate as

$$\bar{z} = x_1 - ix_2 \qquad (4)$$

Adding (3) and (4) gives

$$2x_1 = z + \bar{z} \qquad (5)$$

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Subtracting (4) from (3) gives

$$2ix_2 = z - \bar{z} \tag{6}$$

Expressing (5) in partial differential equation, gives

$$2\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \tag{7}$$

Expressing (6) in partial differential equation, gives

$$2i\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \tag{8}$$

Adding (7) and (8) gives

$$2\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) = 2\frac{\partial}{\partial z} \tag{9}$$

Subtracting (8) from (9) we obtain

$$2\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) = 2\frac{\partial}{\partial \bar{z}} \tag{10}$$

Multiplying (9) and (10) we have

$$4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + 4i\left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_1 \partial x_2}\right) \tag{11}$$

Equating the two right terms of (11) to the right part of (7) and (8) respectively, we obtain

$$4\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) = 4\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) \tag{12}$$

$$4i\left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_1 \partial x_2}\right) = -4i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \tag{13}$$

Replacing the two right hand term of (11) by the two right terms of (12) and (13) respectively, gives

$$4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) - 4i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \tag{14}$$

Let the displacement vectors $u' and\ u''$ be represented in their complex form [12] by

$$w' = u_1' + iu_2', \quad \overline{w'} = u_1' - iu_2' \tag{15}$$

$$w'' = u_1'' + iu_2'', \quad \overline{w''} = u_1'' - iu_2'' \tag{16}$$

Operating (14) on (15) and (16) respectively, we have obtained

$$4\frac{\partial^2 w'}{\partial z \partial \bar{z}} = 4\left(\frac{\partial w'}{\partial z} + \frac{\partial \overline{w'}}{\partial \bar{z}}\right) - 4i\left(\frac{\partial w'}{\partial z} - \frac{\partial \overline{w'}}{\partial \bar{z}}\right) \tag{17}$$

$$4\frac{\partial^2 w''}{\partial z \partial \bar{z}} = 4\left(\frac{\partial w''}{\partial z} + \frac{\partial \overline{w''}}{\partial \bar{z}}\right) - 4i\left(\frac{\partial w''}{\partial z} - \frac{\partial \overline{w''}}{\partial \bar{z}}\right) \tag{18}$$

where the displacement vectors w' and w'' depend on the elastic and plastic regions.

We adopt [12] in order to make (1) solvable.

Let

$$\Delta u' = 4\frac{\partial^2 w'}{\partial z \partial \bar{z}} \quad and \quad \Delta u'' = 4\frac{\partial^2 w''}{\partial z \partial \bar{z}} \tag{19}$$

and

$$\frac{\partial w'}{\partial z} + \frac{\partial \bar{w}'}{\partial \bar{z}} = \frac{\partial\,(u_1' + iu_2')}{\partial\,(x_1 + ix_2)} + \frac{\partial\,(u_1' - iu_2')}{\partial\,(x_1 - ix_2)} = 2div\,u'$$

$$= 2\Theta' \tag{20}$$

Substituting (19) and (20) for $\Delta u'$ and div u' in (1)

we obtain

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$$\left.\begin{array}{l} 4a_1\frac{\partial^2 w'}{\partial z\partial \bar{z}} + 4c\frac{\partial^2 w''}{\partial z\partial \bar{z}} + 2b_1 grad\theta' + 2d grad\theta'' \\ \text{and } 4c\frac{\partial^2 w'}{\partial z\partial \bar{z}} + 4a_2\frac{\partial^2 w''}{\partial z\partial \bar{z}} + 2d grad\theta' + 2b_2 grad\theta'' \end{array}\right\} = \Psi'$$

$$= \Psi'' \tag{21}$$

where our Laplacian here is defined as

$$\Delta = \nabla.\nabla = \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}$$

$$\nabla = grad = \frac{\partial}{\partial \bar{z}} \tag{22}$$

Substituting (22) in (21) we obtain

$$\left.\begin{array}{l} 4a_1\frac{\partial^2 w'}{\partial z\partial \bar{z}} + 4c\frac{\partial^2 w''}{\partial z\partial \bar{z}} + 2b_1\frac{\partial \theta'}{\partial \bar{z}} + 2d\frac{\partial \theta''}{\partial \bar{z}} = \Psi' \\ \text{and } 4c\frac{\partial^2 w'}{\partial z\partial \bar{z}} + 4a_2\frac{\partial^2 w''}{\partial z\partial \bar{z}} + 2d\frac{\partial \theta'}{\partial \bar{z}} + 2b_2\frac{\partial \theta''}{\partial \bar{z}} = \Psi'' \end{array}\right\} \tag{23}$$

$$\frac{\partial}{\partial \bar{z}}\left(4a_1\frac{\partial w'}{\partial z} + 4c\frac{\partial w''}{\partial z} + 2b_1\theta' + 2d\theta''\right) = \Psi' \tag{24}$$

$$\frac{\partial}{\partial \bar{z}}\left(4c\frac{\partial w'}{\partial z} + 4a_2\frac{\partial w''}{\partial z} + 2d\theta' + 2b_2\theta''\right) = \Psi'' \tag{25}$$

From [13], Integrating (24) and (25) wrt $\bar{z}$ we obtain

$$4a_1\frac{\partial w'}{\partial z} + 4c\frac{\partial w''}{\partial z} + 2b_1\theta' + 2d\theta'' = \int \Psi'\, d\bar{z} = \frac{\Psi^*}{\bar{z}} \tag{26}$$

$$4c\frac{\partial w'}{\partial z} + 4a_1\frac{\partial w''}{\partial z} + 2d\theta' + 2b_2\theta'' = \int \Psi''\, d\bar{z} = \frac{\Psi^{**}}{\bar{z}} \tag{27}$$

Where $\frac{\Psi^*}{\bar{z}}$ and $\frac{\Psi^{**}}{\bar{z}}$ are the analytic (non-homogeneous) terms and $\Psi^* = u + iv$ is the displacement function in the transformed state as a result of contact with external force and $\bar{z} = x_1 - ix_2$, is the complex conjugate function.

For the Non-Homogeneous Term $\left(\frac{\Psi^*}{\bar{z}}\right)$

Comparing the non-homogeneous part of (26) and (1), we have that

$$\frac{\Psi^*}{\bar{z}} = \rho F' = \rho(F_1 + iF_2) \tag{28}$$

So that

$$\frac{\Psi^*}{\bar{z}} = \frac{u+iv}{x_1 - i\,x_2} \tag{29}$$

Expressing the above in partial differential equation gives:

$$\frac{\partial \Psi^*}{\partial \bar{z}} = \frac{\partial(u+iv)}{\partial(x_1 - ix_2)} = \frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} + i\frac{\partial v}{\partial x_1} - \frac{\partial v}{\partial x_2}$$

$$= \frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2} + i\frac{\partial u}{\partial x_2} + i\frac{\partial v}{\partial x_1} = \rho F_1 + i\rho F_2 \tag{30}$$

Equating the real and imaginary parts of (30), gives

$$\frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2} = \rho F_1 \tag{31}$$

$$\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} = \rho F_2 \tag{32}$$

We adopt [12] by Introducing new variables φ and η. Let the new variables be given by

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$$u = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \eta}{\partial x_2} \quad (33)$$

$$v = -\frac{\partial \varphi}{\partial x_2} + \frac{\partial \eta}{\partial x_1} \quad (34)$$

Substituting (33) and (34) in (32), we obtain

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \varphi}{\partial x_1} + \frac{\partial \eta}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(-\frac{\partial \varphi}{\partial x_2} + \frac{\partial \eta}{\partial x_1}\right) = \rho F_1 \quad (35)$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial \varphi}{\partial x_1} + \frac{\partial \eta}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(-\frac{\partial \varphi}{\partial x_2} + \frac{\partial \eta}{\partial x_1}\right) = \rho F_2 \quad (36)$$

From (35) and (36) we have that

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} = \nabla^2 \varphi = \rho F_1 \quad (37)$$

$$\frac{\partial^2 \eta}{\partial x_1^2} + \frac{\partial^2 \eta}{\partial x_2^2} = \nabla^2 \eta = \rho F_2 \quad (38)$$

From (37) and (38),

$$\nabla^2 \varphi + i\nabla^2 \eta = \rho F_1 + i\rho F_2 = \rho F$$

$$\nabla^2 (\varphi + i\eta) = \rho (F_1 + iF_2)$$

$$\nabla^2 \varphi = \rho F \qquad \text{(Neglecting the imaginary part)} \quad (39)$$

Newtonian Gravitation and Gravitational Force

In the classical field theory, [5] describes the Newtonian gravitation which describes the gravitational force F, as a mutual interaction between two masses, M_1 and M_2 expressed as:

$$F = -\frac{GM_1M_2}{r^2} \quad (40)$$

In this context, $\mathbf{M_1}$ is the isotropic elastic body (Figure1), $\mathbf{M_2}$ is the object of our forcing term, **G** is the Earth gravitational constant and **r** is the distance between the centre of the two masses $\mathbf{M_1}$ and $\mathbf{M_2}$ respectively. The massive body $\mathbf{M_1}$ has a gravitational field g. Since the gravitational force F, is conservative, the field g can be written as a gradient of a gravitational scalar potential φ as

$$g = -\nabla \varphi \quad (41)$$

Gauss' Law and Poisson Equation for Gravity

Gauss' law of gravity is equivalent to Newton's law of universal gravitation. The differential form of Gauss' law is given as

$$\nabla . g = 4\pi G\rho \quad (42)$$

Where $\nabla . g$ is the divergence, G is the universal constant and ρ is the mass density at each point. Gauss' law is also given in integral form as

$$\oint_{\partial v} g . dA = \int_v \nabla . g dV \quad (43)$$

where V is a closed region bounded by a simple closed oriented surface ∂v which is the infinitesimal piece of the volume and g is the gravitational field. Also, in the case of a gravitational field due to attracting massive objects of density ρ, Gauss' law for gravity in differential form can be used to obtain the corresponding Poisson equation for gravity:

$$\nabla . g = -4\pi G\rho \quad (44)$$

Substituting (41) in (44), gives

$$\nabla(-\nabla \varphi) = -4\pi G\rho$$

$$\nabla^2 \varphi = 4\pi G\rho \quad (45)$$

(45) is the Poisson equation for gravity [11]. Hence, (39) is equivalent to (45) because it involves the mutual interaction between the isotropic elastic body (M_1) in Figure1 and the object of our force term (M_2).

That is

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$$\nabla^2\varphi = \rho F$$

$$\nabla^2\varphi = 4\pi G\rho$$

$$\rho F = 4\pi G\rho$$

$$F = 4\pi G \qquad (46)$$

Hence, our forcing term is a gravitational force, and it is Poisson in nature, as such it is restricted to a plane.

Equation of Equilibrium

We consider the surface area of Figure 1 as follows:

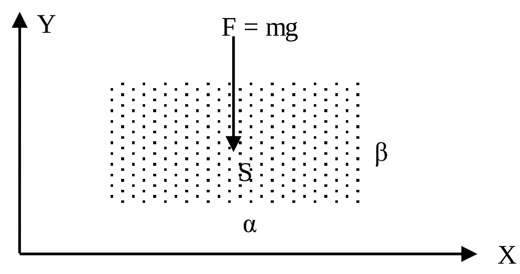

Figure 3: - Surface area of Figure1 and its force distribution

where

F = mg is the gravitational force (mass x gravity)

S = Surface area (length (α) x breadth (β)).

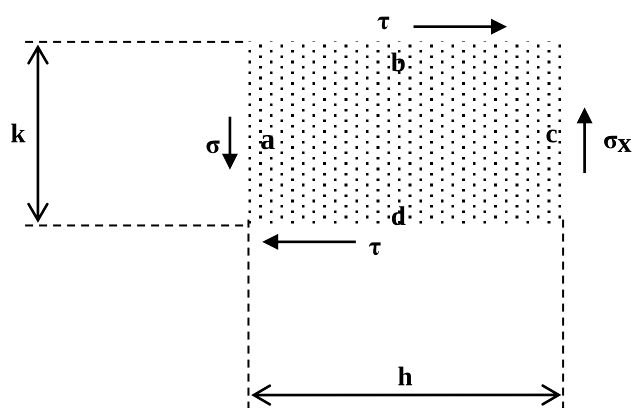

Figure 4: - Stress distribution on a rectangular Plane

where:

σ_x = Normal stress in the x-direction

σ_y = Normal stress in the y-direction

τ_{xy} = Shear stress in the x-direction

τ_{yx} = Shear stress in the y-direction

[6] deduced that

$$\left.\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} = 0 \end{aligned}\right\} \qquad (47)$$

$\tau_{yx} = \tau_{xy}$ (Symmetric).

(48) is the Equation of balance or Equilibrium equation in 2-dimensions.

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Compatibility Equation

Equation (47) shows two equations in three unknowns, with stress components $\sigma_{x,}\ \ \sigma_y\ and\ \tau_{xy}$. For compatibility, we adopt [6] for the strain - displacement relation of the deformation process by introducing

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{48}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \tag{49}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{50}$$

Where u $=$ u$(x, y) and\ v = v(x, y)$are displacement vectors in the transformed state $(x, y - plane)$.

Differentiating (48), (49) and (50) twice with respect to y, x, and xy respectively, we obtain

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2}{\partial y^2} . \frac{\partial u}{\partial x} = \frac{\partial^2 \epsilon_x}{\partial y^2} \tag{51}$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2}{\partial x^2} \frac{\partial v}{\partial y} = \frac{\partial^2 \epsilon_y}{\partial x^2} \tag{52}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \quad \frac{\partial^2}{\partial y^2} . \frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2} . \frac{\partial v}{\partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} 53)$$

From (51), (52) and (53), we obtain

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{54}$$

(54) is the compatibility equation in 2-dimensions.

Biharmonic Equation

To solve (54), we apply the stress – strain relationship [11] for plane stress to obtain

$$\epsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y) \tag{55}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x) \tag{56}$$

$$\gamma_{xy} = \frac{2}{E}(1 + v)\tau_{xy} = \frac{1}{-G}\tau_{xy} = -\frac{\tau_{xy}}{G} \tag{57}$$

where: v = Poisson ratio, E = Young modulus, G = Modulus of rigidity.

Substituting (55), (56) and (57) into (54), we obtain

$$\frac{1}{E}\left[\frac{\partial^2}{\partial y_2}(\sigma_x - v\sigma_y) + \frac{\partial^2}{\partial x_2}(\sigma_y - v\sigma_x)\right] \tag{58}$$

$$= \frac{2}{E}\ (1 + \text{v}) \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \tag{59}$$

$$= 2(1 + \text{v}) \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \tag{60}$$

We differentiate (47) with respect to x and y respectively to eliminate the shearing stress, τ_{xy} and obtain

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \tag{61}$$

$$\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \tag{62}$$

Adding (61) and (62), we obtain

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + 2\ \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \tag{63}$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{1}{2}\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right] \tag{64}$$

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Substituting (64) in (60), gives

$$\frac{\partial^2}{\partial y^2}\left(\sigma_x - v\sigma_y\right) + \frac{\partial^2}{\partial x^2}\left(\sigma_y - v\sigma_x\right) = (1 + v)(-\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial y^2})$$

Such that

$$\frac{\partial^2\sigma_x}{\partial y^2} - v\frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} - v\frac{\partial^2\sigma_x}{\partial x^2} = -\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial y^2} - v\frac{\partial^2\sigma_x}{\partial x^2} - v\frac{\partial^2\sigma_y}{\partial y^2}$$

and

$$\frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} + \frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2} = 0 \qquad (65)$$

To solve (65), we introduce a new function φ called Airy's stress function [6].

For the case under consideration, φ can be defined, such that:

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2},\ \sigma_y = \frac{\partial^2\varphi}{\partial x^2},\ \tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} \qquad (66)$$

Substituting (66) in (65), we have

$$\frac{\partial^2}{\partial y^2}.\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2}{\partial x^2}.\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2}{\partial x^2}.\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2}{\partial y^2}.\frac{\partial^2\varphi}{\partial x^2} = 0 \qquad (67)$$

That is

$$\frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0$$

So that

$$\nabla^2.\nabla^2\varphi = 0$$

$$\nabla^4\varphi = 0 \qquad (68)$$

(68) is called the Biharmonic Equation.

Comparing (46) and (68), our force term, F becomes

$$F = 4\pi G = \nabla^2\varphi = 0$$

Hence, $F = 0$ (69)

(69) shows that the force component is Biharmonic in nature.

Stress State of the Force Term on the Plane

We now consider our force (F), to act on a rectangular plane of area $\alpha\beta$.

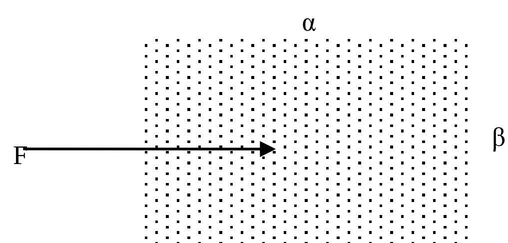

Figure 5: Surface area and force distribution

Generally, stress is the force per unit area of a body/particle and can be expressed as

$$\sigma = \frac{F}{\alpha\beta} \qquad (70)$$

Where σ = stress, Force = force and $\alpha\beta$ = area.

Normal stress (fig. 4) in the x-direction is:

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue VIII, August 2024

$$\sigma_x = a = \frac{F}{\alpha\beta}, \quad (71)$$

Shear stress in the x-direction is given as:

$$\tau_{xy} = b = \frac{F}{\alpha\beta}, \quad (72)$$

While the normal stress in the y-direction is

$$\sigma_y = c = \frac{F}{\alpha\beta} \quad (73)$$

From Airy's stress function φ [6], we have

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} \Rightarrow \varphi = \frac{\sigma_x}{2}y^2 = \frac{a}{2}y^2 + k \quad (74)$$

$$\tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} \Rightarrow \varphi = \tau_{xy}xy = bxy + k \quad (75)$$

$$\sigma_y = \frac{\partial^2\varphi}{\partial x^2} \Rightarrow \varphi = \frac{c}{2}x^2 + \mathrm{k} \quad (76)$$

Adding (74), (75) and (76)

$$\varphi = \frac{a}{2}y^2 + \mathrm{bxy} + \frac{c}{2}x^2 + K \quad (77)$$

(77) is the solution that satisfies a typical Biharmonic Equation in 2-dimensions [6].

Stress Distribution on the X and Y Coordinates

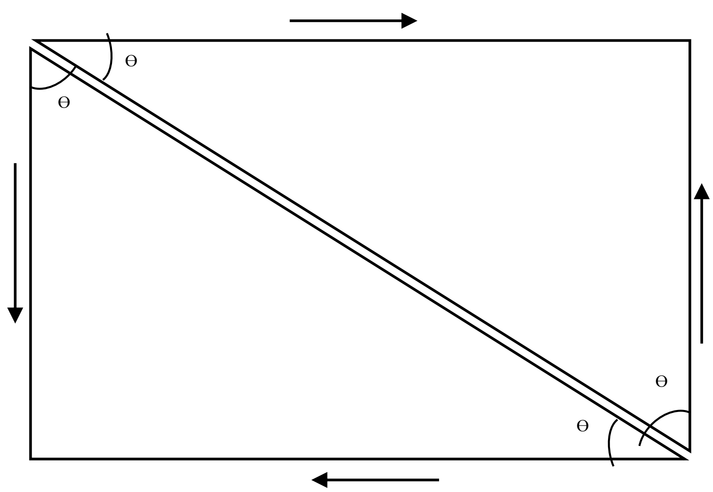

Figure 6: Stress distributions in a 2-dimensional rectangular plane

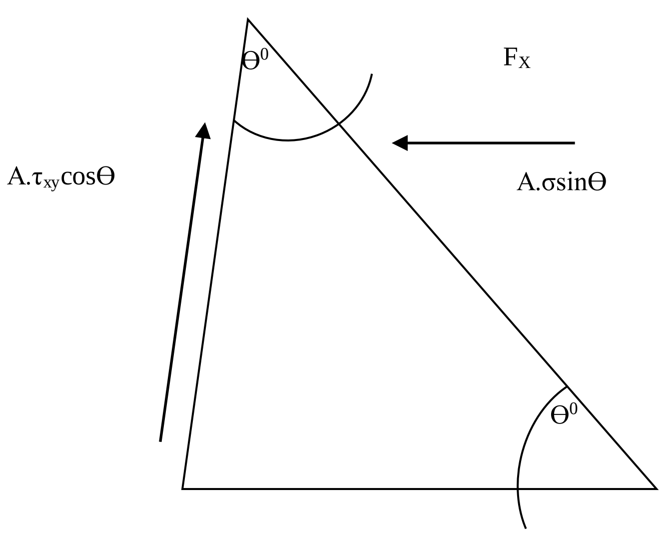

Figure 7: Stress distribution on the x- direction

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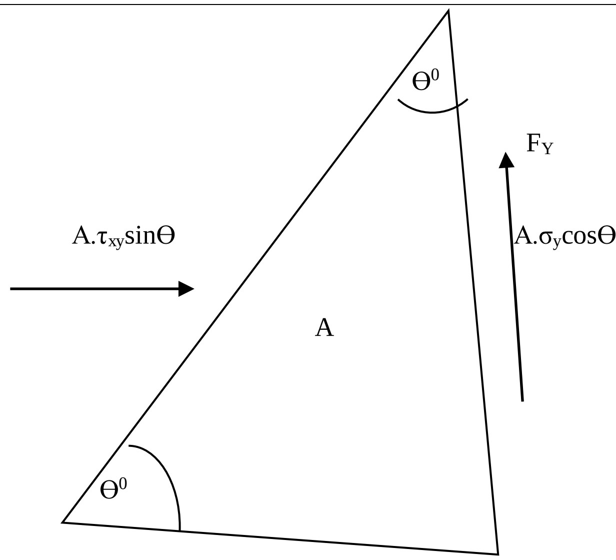

Figure 8: Stress distribution on the y – direction

Deducing from figure 7, the force impact (stress) on the particle in the x-coordinate is obtained as

$$F_x = A\,(\tau_{xy}\cos\Theta + \sigma_x\sin\Theta) \quad (78)$$

$$\Longrightarrow \frac{F_X}{A} = \tau_{xy}\cos\theta + \sigma_x\sin\theta \quad (79)$$

Similarly, from Figure 8, the force impact (stress) on the particle in the y – coordinate is as

$$F_Y = A(\tau_{xy}\sin\theta + \sigma_y\cos\theta) \quad (80)$$

$$\Longrightarrow \frac{F_Y}{A} = \tau_{xy}\sin\theta + \sigma_y\cos\theta \quad (81)$$

Hence, we have the system:

$$\frac{F_X}{A} + \frac{F_Y}{A} = \frac{F}{A} = \begin{pmatrix} \tau_{xy}\cos\theta & \sigma_x\sin\theta \\ \tau_{xy}\sin\theta & \sigma_y\cos\theta \end{pmatrix} \quad (82)$$

(82) is the stress impact on the coordinates.

Boundary Condition of the Force/stress on the Plane

The boundary condition is obtained from the requirement that the total stress on the planes of the particle is zero when no force was introduced. That is, the magnitude of the total stress component, $\det\left(\frac{F}{A}\right) = 0$

$$\Longrightarrow \begin{vmatrix} \tau_{xy}\cos\theta & \sigma_x\sin\theta \\ \tau_{xy}\sin\theta & \sigma_y\cos\theta \end{vmatrix} = 0 \quad (83)$$

Then, $\tau_{xy}\cos\theta\,.\,\sigma_y\cos\theta - \sigma_x\sin\theta.\tau_{xy}\sin\theta = 0$

$$\sigma_y\cos^2\theta = \sigma_x\sin^2\theta$$

$$\frac{\sigma_y}{\sigma_x} = \tan^2\theta \quad (84)$$

(84) is our third result showing the magnitude of the ratio of the normal stressese on the particle.

III. Results

From (68) and (69), the magnitude of the force components on the particle is zero at equilibrium.

From [10], the magnitude of the force term is given by

$F = s\,(\cos^2 a - \sin^2 a)$.

From (84), our graph is given by:

ISSN 2278-2540 | DOI: 10.51583/IJLTEMAS | Volume XIII, Issue VIII, August 2024

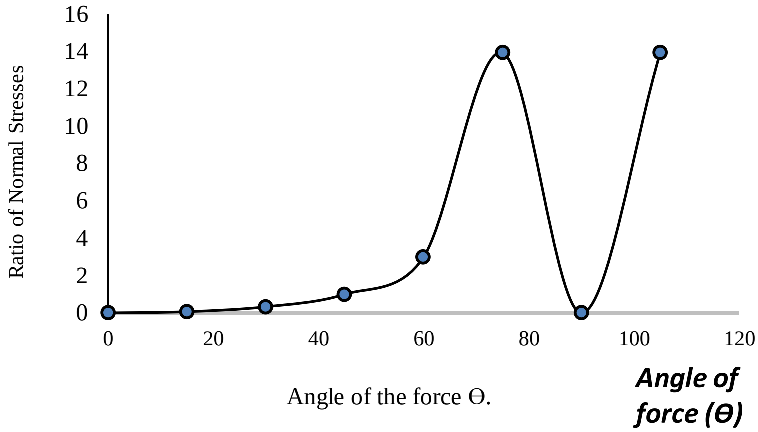

Figure 9: Graph of the ratio of normal stresses against the angle Θ.

IV. Discussion of Results

The graph in Figure 9 above results from the relationship between the ratio of the normal forces and the angle of the force. The curve rises from the origin, 0 when no force term was introduced and rises uniformly to the point when the force acted at angle 60^o and increases rapidly when the force acted above 60^o. The magnitude of the ratio of the normal stresses on the particle attained its maximum value when the force acted at angle 75^o. Beyond 75^o, the particle demonstrates its elastic nature and the curve drops rapidly downward to the origin, 0.When the force acted at 90^0on the plane of the particle, the body regains its elastic potential and returns to its equilibrium state as the stress resolves itself to zero due to the perfect angular formation which allows even distribution of the stresses on both coordinates. When the force acted beyond 90^o, the magnitude of the ratio of the normal stresses on the particle oscillates back to its maximum point, which shows that the elastic potential of the particle has been weakened.

V. Conclusion

In this paper, the problem of non-homogeneous equation of statics in the theory of elastic mixture was considered using complex variable theory. Our theoretical solution for the stress state of the isotropic elastic body examined was found to be consistent with the experimental existing result.

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